PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 20 (34) 1976, pp. 91–98

SOME PROPERTIES OF WRIGHT'S FUNCTION

Lj. Gajić and B. Stanković

(Communicated February 26, 1976.)

The function of E. M. Wright [3]:

$$\Phi(\beta; -\sigma; z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n+1)\Gamma(\beta - n\sigma)}, \qquad 0 < \sigma < 1,$$

β complex number, was found to be very useful in the theory and applications of Mikusiński operators. Even if many properties of this function have been proved [2], [3], we shall prove some new ones which we need in the theory of operator differential equations.

The following two propositions show that the properties, proved in case z a real number [2], can be enlarged for z complex.

Proposition 1. *Let us suppose that* λ *is a complex number,* $\lambda = e^{\alpha i}$, *and that for* α, β, x_0 *of the following two restrictions, is satisfied:*

1. $|\alpha| < \frac{\pi}{2}(1-\sigma)$, β *arbitrary real number,* $x_0 > 0$;

2. $\alpha = \pm\frac{\pi}{2}(1-\sigma)$, $\beta > 0$, $x_0 > 0$;

then:

$$x^{\beta-1}\Phi(\beta; -\sigma; -\lambda x^{-\sigma}) = \frac{1}{2\pi i}\int_{x_9 - i\infty}^{x_0 + i\infty} s^{-\beta} e^{xs - \lambda s^{\sigma}} ds, \quad x > 0. \tag{1}$$

If we require in addition to the restriction 1 *or* 2 *also that* $\beta < 1$ *we can take* $x_0 = 0$ *and then:*

$$x^{\beta-1}\Phi(\beta; -\sigma; -\lambda x^{-\sigma}) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} s^{-\beta} e^{xs - \lambda s^{\sigma}} ds, \quad x > 0.$$

Proof. Let us start from

$$\int_{C'} s^{-\beta} e^{xs - \lambda s^{\sigma}} ds = 0. \tag{2}$$

Where C' is the closed contour given in figure 1.

This integral equals zero because the subintegral function is regular inside and on the contour C'. We shall decompose it:

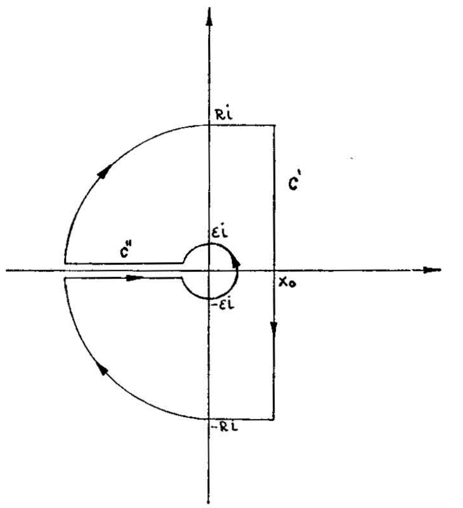

Fig. 1

$$\int_{C''} s^{-\beta} e^{xs-\lambda s^\sigma} ds + \int_{\pi}^{\frac{\pi}{2}} R^{-\beta} e^{-\beta ti + xRe^{ti} - R^\sigma e^{\sigma ti}} Rie^{ti} dt + \int_0^{x_0} (z+Ri)^{-\beta} e^{x(z+Ri)-\lambda(z+Ri)^\sigma} dz + \int_{x_0-i\infty}^{x_0+i\infty} s^{-\beta} e^{xs-\lambda s^\sigma} ds + \int_{-\frac{\pi}{2}}^{-\pi} R^{-\beta} e^{-\beta ti + xRe^{ti} - \lambda R^\sigma e^{\sigma ti}} Rie^{ti} dt + \int_{x_0}^{0} (z-Ri)^{-\beta} e^{x(z-Ri)-\lambda(z-Ri)^\sigma} dz = 0.$$

When $R\to\infty$, some of these integrals tend to zero. For the second and fifth the proof is the same. We shall give it only for the second one.

$$\left|\int_{\pi}^{\frac{\pi}{2}} R^{1-\beta} e^{-\beta ti + xRe^{ti} - \lambda R^\sigma e^{\sigma ti}} ie^{ti} dt\right| \leqslant \int_{\pi}^{\frac{\pi}{2}+\varepsilon} R^{1-\beta} e^{R(x\cos t - R^{\sigma-1}\cos(\alpha+\sigma t))} dt + \int_{\frac{\pi}{2}+\varepsilon}^{\frac{\pi}{2}} R^{1-\beta} e^{R(x\cos t - R^{\sigma-1}\cos(\alpha+\sigma t))} dt = \mathrm{I}_1 + \mathrm{I}_2.$$

It is easy to see that $\lim_{R\to\infty} I_1 = 0$, if it is supposed that the restriction 1 or 2 is satisfied.

Let us consider now I_2 with the restriction 1. We suppose that the number ω is defined in such a way that

$$\alpha = \pm\left(\frac{\pi}{2}(1-\sigma)-\omega\right) \text{ and for } \varepsilon>0, \quad 0<\varepsilon<\frac{\omega}{\sigma},$$

then we have $\frac{\pi}{2} - \omega \leqslant \sigma t + \alpha \leqslant \frac{\pi}{2} + \sigma\varepsilon - \omega$ or

$$-\left(\frac{\pi}{2}(1-\sigma)-\omega\right) + \frac{\sigma\pi}{2} \leqslant \sigma t + \alpha \leqslant -\left(\frac{\pi}{2}(1-\sigma)-\omega\right) + \frac{\sigma\pi}{2} + \sigma\varepsilon.$$

Consequently, $\cos(\sigma t+\alpha)>0$ when $\sigma t+\alpha$ remains in the given intervals and $\lim_{R\to\infty} I_2=0$.

Let us suppose that restriction 2 is satisfied and $\alpha=\frac{\pi}{2}(1-\sigma)$. Taking care of the following inequalities:

$$\cos t\leqslant 1-\frac{2}{\pi}t,\quad \frac{\pi}{2}\leqslant t\leqslant\frac{\pi}{2}+\varepsilon \quad\text{and}\quad \cos(\alpha+\sigma t)\geqslant\frac{\pi}{2}-t,\quad \frac{\pi}{2}\leqslant t\leqslant\frac{\pi}{2}+\varepsilon,$$

$\frac{\pi}{2}\leqslant\alpha+\sigma t\leqslant\frac{\pi}{2}+\sigma\varepsilon$ we have:

$$I_2=\int_{\frac{\pi}{2}+\varepsilon}^{\frac{\pi}{2}} R^{1-\beta}e^{xR\cos t-R^{\sigma}\cos(\alpha+\sigma t)}dt\leqslant$$

$$\leqslant\int_{\frac{\pi}{2}+\varepsilon}^{\frac{\pi}{2}} R^{1-\beta}\exp\left(xR\left(1-\frac{2}{\pi}t\right)-R^{\sigma}\left(\frac{\pi}{2}-t\right)\right)dt\leqslant$$

$$\leqslant R^{1-\beta}\int_{\frac{\pi}{2}+\varepsilon}^{\frac{\pi}{2}}\exp\left(xR-\frac{\pi}{2}R^{\sigma}-\left(\frac{2xR}{\pi}-R^{\sigma}\right)t\right)dt=0\,(R^{-\beta}).$$

The same treatment applies in case $\alpha=-\frac{\pi}{2}(1-\sigma)$.

As the proof for the third and sixth integral is the same, we give it only for the third one:

$$\left|\int_0^{x_0}(t+Ri)^{-\beta}e^{x(t+Ri)-\lambda(t+Ri)^{\sigma}}dt\right|\leqslant$$

$$\leqslant\int_0^{x_0}(t^2+R^2)^{-\frac{\beta}{2}}\exp\left(xt-(t^2+R^2)^{\frac{\sigma}{2}}\cos\left(\alpha+\sigma\frac{\pi}{2}-\delta\sigma\right)\right)dt.$$

Where δ depends on t and $0\leqslant\delta\leqslant\delta_0$, $\delta_0=\frac{\pi}{2}-\arg(x_0+iR)\to 0$, when $R\to\infty$.

Let us suppose that the restriction 1 is satisfied and that $\alpha=\pm\left(\frac{\pi}{2}(1-\sigma)-\omega\right)$, then $\alpha+\frac{\sigma\pi}{2}-\delta\sigma=\frac{\pi}{2}-\omega-\sigma\delta$ or $\alpha+\frac{\sigma\pi}{2}-\delta\sigma=-\left(\frac{\pi}{2}(1-\sigma)-\omega\right)+\frac{\sigma\pi}{2}-\sigma\delta$; consequently $\cos\left(\alpha+\frac{\sigma\pi}{2}-\sigma\delta\right)>0$, $0\leqslant\delta\leqslant\delta_0$ and the last integral tends to zero when $R\to\infty$.

With restriction 2, $\cos\left(\alpha+\frac{\sigma\pi}{2}-\sigma\delta\right)\geqslant 0$ and

$$\int_0^{x_0}(t^2+R^2)^{-\frac{\beta}{2}}e^{xt}\exp\left(-(t^2+R^2)^{\frac{\sigma}{2}}\cos\left(\alpha+\frac{\sigma\pi}{2}-\sigma\delta\right)\right)dt=0\,(R^{-\beta}).$$

After this, there remains only

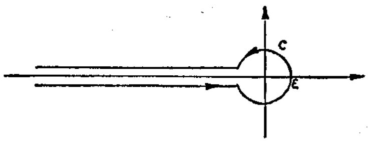

Fig. 2

$$\frac{1}{2\pi i}\int_C s^{-\beta}e^{xs-\lambda s^\sigma}ds=\tag{3}$$
$$=\frac{1}{2\pi i}\int_{x_0-i\infty}^{x_0+i\infty}s^{-\beta}e^{xs-\lambda s^\sigma}ds,$$

where C is contour in figure 2. We know ([2] p. 114)

$$\frac{1}{2\pi i}\int_C s^{-\beta}e^{s+zs^\sigma}ds=\Phi(\beta;\ -\sigma;\ z),$$

z is complex number. If we introduce the change $z=-\lambda x^{-\sigma}$ and $\frac{s}{x}=v$, $x\neq 0$ we have

$$x^{\beta-1}\Phi(\beta;\ -\sigma;\ -\lambda x^{-\sigma})=\frac{1}{2\pi i}\int_C v^{-\beta}e^{vx-\lambda v^\sigma}dv.\tag{4}$$

From (3) and (4) follows the requested relation (1).

We shall introduce our additional restriction, namely $\beta<1$. Now we shall start from the integral

$$\int_{C'''}s^{-\beta}e^{xs-\lambda s^\sigma}ds=0.$$

Where C''' is the closed contour in figure 3. Supposing the restriction 1 or 2 satisfied, the integrals over the parts A and B tend to zero and

$$\int_{x_0-i\infty}^{x_0+i\infty}s^{-\beta}e^{xs-\lambda s^\sigma}ds=\int_{-i\infty}^{i\varepsilon}s^{-\beta}e^{xs-\lambda s^\sigma}ds+$$
$$+\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\varepsilon^{-\beta}e^{-\beta ti+x\varepsilon e^{ti}-\lambda\varepsilon^\sigma e^{\sigma ti}}\varepsilon ie^{ti}\,dt+$$
$$+\int_{i\varepsilon}^{i\infty}s^{-\beta}e^{xs-\lambda s^\sigma}ds.$$

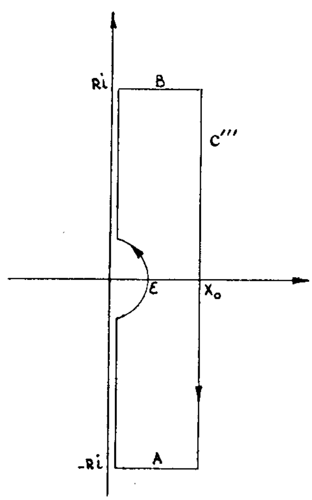

Fig. 3

With our additional restriction, $\beta<1$:

$$\left|\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \varepsilon^{-\beta+1} e^{-\beta ti+x\varepsilon e^{ti}-\lambda\varepsilon^{\sigma}e^{\sigma ti}} ie^{ti}\,dt\right| \leqslant \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \varepsilon^{1-\beta} e^{x\varepsilon\cos t-\varepsilon^{\sigma}\cos(\alpha+\sigma t)}\,dt \to 0, \qquad \varepsilon\to 0.$$

and our relation (2) is proved.

Proposition 2. *If* $0<\sigma<1$, $|\alpha|<\frac{\pi}{2}(1-\sigma)$ *and* $\beta<1$, *then*

$$|x^{\beta-1}\Phi(\beta;\,-\sigma;\,-\lambda x^{-\sigma})|\leqslant\frac{C}{2\pi\sigma}\Gamma\left(\frac{1-\beta}{\sigma}\right),\qquad x>0 \tag{5}$$

where

$$C=\cos^{\frac{\beta-1}{\sigma}}\left(\alpha+\frac{\sigma\pi}{2}\right)+\cos^{\frac{\beta-1}{\sigma}}\left(\alpha-\frac{\sigma\pi}{2}\right) \tag{6}$$

and

$$x^{\beta-1}\Phi(\beta;\,-\sigma;\,-\lambda x^{-\sigma})=0\,(x^{\beta-1}),\qquad x\to\infty. \tag{7}$$

Proof. Let us suppose that $0<\delta\leqslant x\leqslant M$, δ and M fixed. Using relation (2) we have:

$$|x^{\beta-1}\Phi(\beta;\,-\sigma;\,-\lambda x^{-\sigma})|\leqslant\frac{1}{2\pi}\int_{-i\infty}^{i\infty}|s|^{-\sigma}e^{Re(xs-\lambda s^{\sigma})}|\,ds|\leqslant \tag{8}$$

$$\leqslant\frac{1}{2\pi}\int_0^{\infty}y^{-\beta}\exp\left(-y^{\sigma}\cos\left(\alpha+\frac{\sigma\pi}{2}\right)\right)dy+\frac{1}{2\pi}\int_0^{\infty}y^{-\beta}\exp\left(-y^{\sigma}\cos\left(\alpha-\frac{\sigma\pi}{2}\right)\right)dy\leqslant$$

$$\leqslant\frac{1}{2\pi\sigma}\left(\cos^{\frac{\beta-1}{\sigma}}\left(\alpha+\frac{\sigma\pi}{2}\right)+\cos^{\frac{\beta-1}{\sigma}}\left(\alpha-\frac{\sigma\pi}{2}\right)\right)\int_0^{\infty}t^{\frac{-\beta+1}{\sigma}-1}e^{-t}\,dt=\frac{C}{2\pi\sigma}\Gamma\left(\tfrac{1-\beta}{\sigma}\right).$$

This inequality holds because $\cos\left(\alpha\pm\sigma\frac{\pi}{2}\right)\neq 0$ for $|\alpha|<\frac{\pi}{2}(1-\sigma)$.

We shall now consider our function in the neighbourhood of $x=0$. If $|\arg(\lambda x^{-\sigma})|\leqslant\min\left(\frac{3\pi}{2}(1-\sigma),\,\pi\right)-\varepsilon$ then ([2] p. 121):

$$|x^{1-\beta}\Phi(\beta:\,-\sigma;\,-\lambda x^{-\sigma})|\leqslant Cx^{1-\beta-\left(\frac{1}{2}-\beta\right)\frac{\sigma}{1-\sigma}}\sigma^{\sigma\left(\frac{1}{2}-\beta\right)\cdot\frac{1}{1-\sigma}}(1-\sigma)^{\frac{1}{2}-\beta}\cdot$$

$$\cdot\exp\left(-(1-\sigma)^{\frac{\sigma}{1-\sigma}}x^{-\frac{\sigma}{1-\sigma}}\cos\frac{\alpha}{1-\sigma}\right).$$

This inequality shows that our function Φ tends to zero very fast when $x\to 0^+$ and it is possible to choose δ in such a way that the inequality (5) is true for all $0\leqslant x\leqslant M$.

We have to analyze the behaviour of the function Φ when $x\to\infty$. Using relation (4):

$$|x^{\beta-1}\Phi(\beta;-\sigma;-\lambda x^{-\sigma})|=\left|\frac{1}{2\pi i}\int_C s^{-\beta}e^{xs-\lambda s^{\sigma}}ds\right|\leqslant$$

$$\leqslant\frac{x^{\beta-1}}{2\pi}\left(\int_{\varepsilon}^{\infty}t^{-\beta}e^{-t-x^{-\sigma}t^{\sigma}\cos(\alpha-\sigma\pi)}dt+\right.$$

$$+\int_{-\pi}^{\pi}\varepsilon^{-\beta+1}e^{\varepsilon\cos t-x^{-\sigma}\varepsilon^{\sigma}\cos(\alpha+\sigma t)}|dt|+$$

$$\left.+\int_{\varepsilon}^{\infty}t^{-\beta}e^{-t-x^{-\sigma}t^{\sigma}\cos(\alpha+\sigma t)}dt\right).$$

Let us consider each of these theree integrals

$$\varepsilon^{1-\beta}\int_{-\pi}^{\pi}e^{\varepsilon\cos t-x^{-\sigma}\varepsilon^{\sigma}\cos(\alpha+\sigma t)}|dt|\leqslant\varepsilon^{1-\beta}\int_{-\pi}^{\pi}e^{\varepsilon+\varepsilon^{\sigma}x^{-\sigma}}|dt|\to 0,\qquad \varepsilon\to 0,$$

$$\frac{x^{\beta-1}}{2\pi}\int_0^{\infty}t^{-\beta}\exp\left(-t-\left(\frac{t}{x}\right)^{\sigma}\cos(\alpha-\sigma\pi)\right)dt=$$

$$=\frac{1}{2\pi}\int_0^{\infty}v^{-\beta}e^{-xv}e^{-v^{\sigma}\cos(\alpha-\sigma\pi)}dv\sim\frac{1}{2\pi}\,\frac{\Gamma(1-\beta)}{x^{1-\beta}}\to 0$$

when $x\to\infty$ (see [1]).

The third proposition contains the behaviour of the function $x^{-1}\Phi(0;-\sigma;-\lambda x^{-\sigma})$ in the right neighbourhood of zero. The difficulty lies in the fact that this function has an essential singularity in $x=0$ and tends exponentially to zero when $x\to 0^{+}$. For this reason we find first an analytical continuation by a suitable function.

Proposition 3. *Let* $C(\alpha, k)$ *be*

$$C(\alpha,k)=\begin{cases}\dfrac{\Gamma\left(\dfrac{k+1}{\sigma}\right)}{\sigma\pi\,\Gamma(k+1)}\cdot\left(\cos^{-\frac{(k+1)}{\sigma}}\left(\sigma\dfrac{\pi}{2}-\alpha\right)+\cos^{-\frac{(k+1)}{\sigma}}\left(\sigma\dfrac{\pi}{2}+\alpha\right)\right), & \alpha\neq 0,\\[2ex] \dfrac{\Gamma\left(\dfrac{k+1}{\sigma}\right)}{\sigma\pi\,\Gamma(k+1)}\cdot\left(\cos^{-\frac{(k+1)}{\sigma}}\left(\sigma\dfrac{\pi}{2}\right)\right), & \alpha=0,\end{cases}$$

then for $0<\sigma<1$; $\lambda=e^{\alpha i}$, $|\alpha|<\frac{\pi}{2}(1-\sigma)$ *and* $0\leqslant x\leqslant T$: $x^{-1}\Phi(0;\,-\sigma;\,-\lambda x^{-\sigma})\approx$ ≈ 0 *and*

$$|x^{-1}\Phi(0;\,-\sigma;\,-\lambda x^{-\sigma})|\leqslant C(\alpha,2n)\,T^{2n}+C(\alpha,2n+1)\,T^{2n+1}.$$

Proof. From the proposition 1, relation (2) follows

$$x^{-1}\Phi(0;\,-\sigma;\,-\lambda x^{-\sigma})=$$

$$=\frac{1}{2\pi}\int_0^\infty \exp\left(-t^\sigma\cos\left(\alpha-\sigma\frac{\pi}{2}\right)\right)\cos\left(xt+t^\sigma\sin\left(\alpha-\sigma\frac{\pi}{2}\right)\right)dt-$$

$$-\frac{i}{2\pi}\int_0^\infty \exp\left(-t^\sigma\cos\left(\alpha-\sigma\frac{\pi}{2}\right)\right)\sin\left(xt+t^\sigma\sin\left(\alpha-\sigma\frac{\pi}{2}\right)\right)dt+$$

$$+\frac{1}{2\pi}\int_0^\infty \exp\left(-t^\sigma\cos\left(\alpha-\sigma\frac{\pi}{2}\right)\right)\cos\left(xt-t^\sigma\sin\left(\alpha+\sigma\frac{\pi}{2}\right)\right)dt+$$

$$+\frac{i}{2\pi}\int_0^\infty \exp\left(-t^\sigma\cos\left(\alpha+\sigma\frac{\pi}{2}\right)\right)\sin\left(xt-t^\sigma\sin\left(\alpha+\sigma\frac{\pi}{2}\right)\right)dt=$$

$$=\frac{1}{2\pi}\int_0^\infty \exp(-t^\sigma\cos p)(\cos xt\cdot\cos(t^\sigma\sin p)-\sin xt\cdot\sin(t^\sigma\sin p))\,dt-$$

$$-\frac{i}{2\pi}\int_0^\infty \exp(-t^\sigma\cos p)(\sin xt\cdot\cos(t^\sigma\sin p)+\cos xt\cdot\sin(t^\sigma\sin p))\,dt+$$

$$+\frac{1}{2\pi}\int_0^\infty \exp(-t^\sigma\cos q)(\cos xt\cos(t^\sigma\sin q)+\sin xt\sin(t^\sigma\sin q))\,dt+$$

$$+\frac{i}{2\pi}\int_0^\infty \exp(-t^\sigma\cos q)(\sin xt\cos(t^\sigma\sin q)-\cos xt\sin(t^\sigma\sin q))\,dt=$$

$$=I_1(p)-I_2(p)-iI_3(p)-iI_4(p)-I_1(q)+I_2(q)+iI_3(q)-iI_4(q)$$

where $p=\alpha-\sigma\frac{\pi}{2}$ and $q=\alpha+\sigma\frac{\pi}{2}$.

We shall find Taylor polynomial and remainder for every integral I_k. So for $I_1(p)$ we have

$$I_1(p)=\frac{1}{2\pi}\int_0^\infty \exp(-t^\sigma\cos p)\cos xt\cos(t^\sigma\sin p)\,dt=$$

$$=\frac{1}{2\pi\sigma}\sum_{k=0}^\infty\frac{(-1)^k x^{2k}}{(2k)!}\Gamma\left(\frac{2k+1}{\sigma}\right)\cos p\left(\frac{2k+1}{\sigma}\right)+R_1(n,p).$$

In such a way we can express all the integrals I_k. From the sum of every two integrals with the same indices remains only the sum of remainders: $I_k(p)+I_k(q)=R_k(n,p)+R_k(n,q)$, $k=1, 2, 3, 4$. We shall show this only for $k=1$.

$$I_1(p)+I_1(q)=\frac{1}{2\pi\sigma}\sum_{k=0}^\infty\frac{(-1)^k}{(2k)!}x^{2k}\Gamma\left(\frac{2k+1}{\sigma}\right)\left(\cos p\left(\frac{2k+1}{\sigma}\right)+\cos q\left(\frac{2k+1}{\sigma}\right)\right)+$$

$$+R_1(n,p)+R_1(n,q)=$$

$$=\frac{1}{2\pi\sigma}\sum_{k=0}^\infty\frac{(-1)^k}{(2k)!}x^{2k}\Gamma\left(\frac{2k+1}{\sigma}\right)2\cos\left(\frac{2k+1}{\sigma}\right)\left(\frac{p+q}{2}\right)\cos\left(\frac{2k+1}{\sigma}\right)\left(\frac{p-q}{2}\right)+$$

$$+R_1(n,p)+R_1(n,q).$$

Now is easy to find an estimate for the rests. As the treatement is the same for all the rests we shall give it only for $R_1(n,p)$

$$\left|R_1(n,p)\right|=\left|\frac{1}{2\pi}\int_0^\infty e^{-t^\sigma\cos p}\cos(t^\sigma\sin p)\frac{(xt)^{2n}}{(2n)!}(\cos(\theta xt))^{(2n)}dt\right|\leqslant$$

$$\leqslant\frac{x^{2n}}{2\pi(2n)!}\int_0^\infty e^{-t^\sigma\cos p}t^{2n}dt\leqslant\frac{\Gamma\left(\frac{2n+1}{\sigma}\right)}{2\pi\sigma\Gamma(2n+1)\cos^{(2n+1)/\sigma}p}x^{2n}.$$

REFERENCES

[1] G. Doetsch, *Handbuch der Laplace Transformation*, Band I, Birkhäuser Verlag, Basel (1950) s. 473.

[2] B. Stanković, *On the function of E. M. Wright*, Publications de l'Institut Mathématique. T. 10 (24), (1970), pp. 113—124.

[3] E. M. Wright, *The generalized Bessel function of order greater than one*, Quart. J. Math. Oxford series, 2 (1940) 36—48.