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FIXED POINTS OF WEAKLY CONTRACTION MAPPINGS

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Introduction. Let (M, d) be a metric space and T a selfmapping of M into itself. A mapping T is called a *contraction* if

$$d(Tx, Ty) \leqslant \alpha \cdot d(x, y) \tag{1}$$

holds for some $\alpha < 1$ and all $x, y \in M$. A well-known theorem of Banach states that if M is complete and T is a contraction on M, then T has a unique fixed point u in M and for each x in M $\lim_n T^n x = u$ and $d(T^n x, u) \leqslant \frac{\alpha^n}{1-\alpha} d(x, Tx)$.

E. Rekotch [7] has proved that if $\alpha = \alpha(d(x, y))$ is a monotonically decreasing function and $0 \leqslant \alpha(t) < 1$ for $t > 0$, then a mapping satisfying (1) has a unique fixed point. D. Boyd and J. Wong [1] and F. Browder [2] extended some of results of [7]. A. Meir and E. Keeler [6] have introduced and investigated mappings which satisfy the following condition:

Given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\varepsilon \leqslant d(x, y) < \varepsilon + \delta \quad \text{implies} \quad d(Tx, Ty) < \varepsilon. \tag{2}$$

Such mappings they called *weakly uniformly strict contractions*. For these mappings Meir and Keeler have proved a fixed point theorem which is an extension of the results of [7] and [1].

In [3] (cf. [4]) a notion of a generalized contraction was introduced as follows:

A mapping $T: M \to M$ is called a *generalized contraction* if

$$d(Tx, Ty) \leqslant \alpha \cdot \max\left\{d(x, y),\ d(x, Tx),\ d(y, Ty),\ \frac{1}{2}[d(x, Ty) + d(y, Tx)]\right\} \tag{3}$$

holds for some $\alpha < 1$ and all $x, y \in M$.

In this paper we introduce and investigate mappings which satisfy the following condition:

For each $\varepsilon>0$, there exists $\delta(\varepsilon)>0$ such that

$$(4)\qquad \varepsilon\leqslant\max\left\{d(x, y),\ d(x, Tx),\ d(y, Ty),\ \frac{1}{2}[d(x, Ty)+d(y, Tx)]\right\}<\varepsilon+\delta(\varepsilon)$$

implies

$$d(Tx, Ty)<\varepsilon.$$

We present a fixed point theorem with an estimate of a distance between a fixed point and its approximation. This result is an extension of some results of [3] and the results of [1], [6], [7]. and others.

1. Let $T: M\rightarrow M$ be a mapping of a metric space M into itself. Recall that T is said to be *orbitally continuous* if $\lim_{i\rightarrow\infty} T^{n_i}x=u$ implies $\lim_{i\rightarrow\infty} TT^{n_i}x=Tu$ and that M is said to be *T-orbitally complete* if every Cauchy sequence of the form $\{T^{n_i}x\}_{i=1}^{\infty}$ converges in M.

Before we shall state our result we observe that (4) and

$$\max\left\{d(x, y),\ d(x, Tx),\ d(y, Ty),\ \frac{1}{2}[d(x, Ty)+d(y, Tx)]\right\}>0$$

immediately imply

$$(5)\qquad d(Tx, Ty)<\max\left\{d(x, y),\ d(x, Tx),\ d(y, Ty),\ \frac{1}{2}[d(x, y),\ d(y, Tx)]\right\}.$$

Note that the condition (4) does not guarantee the existence of a fixed point of T, even if M is compact and connected.

Example 1. Let $M=[0, 2]$ be a subset of reals with usual metric and let a function T of M into itself be befined by $Tx=\frac{x}{2}$ for $x\neq 0$ and $T(0)=1$. Since $d(Tx, Ty)=\frac{1}{2}d(x, y)$ for $x\cdot y\neq 0$ and $d(Tx, T0)=1-\frac{x}{2}$ and

$$\max\left\{d(x, 0),\ d(x, Tx),\ d(0, T0),\ \frac{1}{2}[d(x, T0)+d(0, Tx)]\right\}\geqslant d(0, T0)=1,$$

we conclude that (5) is satisfied and $d(Tx, Ty)<1$ for all $x, y\in M$. Also T satisfies (4) with $\delta(\varepsilon)=\min\left\{\varepsilon\frac{1-\varepsilon}{2}\right\}$ for $\varepsilon<1$ and $\delta(\varepsilon)=1$ for $\varepsilon\geqslant 1$. Indeed, for $\varepsilon<1$ (4) is satisfied, since then must be $x\cdot y\neq 0$ (as $\varepsilon<1$ implies $\varepsilon+\delta(\varepsilon)<1$), and for $\varepsilon\geqslant 1$ (4) is satisfied because $d(Tx, Ty)<1$ always. Note that here T is not orbitally continuous.

Now we are going to prove some fixed point theorems for maps which satisfy (4).

Theorem 1. *Let (M, d) be a metric space and T a selfmapping of M satisfying* (4). *Then for each x in M $\{T^n x\}_{n=0}^{\infty}$ is a Cauchy sequence. Furthermore, if M is T-orbitally complete and T in addition satisfies the condition*

$$(4') \quad d(Tx, Ty) \leqslant \max\left\{d(x, y),\ \alpha[d(x, Tx) + d(y, Ty)],\ \frac{1}{2}[d(x, Ty) + d(y, Tx)]\right\}$$

for some α, $0 < \alpha < 1$, or T is orbitally continuous, then

a) *There exists a unique fixed point u of T,*

b) $\lim\limits_{n\to\infty} T^n x = u$ *and*

c) $d(T^{n-1}x, T^n x) < \min\{\varepsilon, \delta(\varepsilon)\}$ *implies* $d(T^n x, u) < \varepsilon$ *for every* $x \in M$.

Proof. For $x, y \in M$ denote

$$m(x, y, T) = \max\left\{d(x, y),\ d(x, Tx),\ d(y, Ty),\ \frac{1}{2}[d(x, Ty) + d(y, Tx)]\right\}.$$

First we shall show that (4) implies

$$(6) \qquad m(x, Tx, T) = d(x, Tx)$$

for each $x \in M$. If $m(x, Tx, T) = 0$, then (6) trivially holds. Assume now that $m(x, Tx, T) > 0$. Then by (5)

$$d(Tx, TTx) < m(x, Tx, T) = \max\left\{d(x, Tx),\ d(Tx, TTx),\ \frac{1}{2} d(x, TTx)\right\}.$$

As $d(Tx, TTx) < d(Tx, TTx)$ is absurd, it follows that

$$m(x, Tx, T) = \max\left\{d(x, Tx),\ \frac{1}{2} d(x, TTx)\right\}.$$

Suppose that $d(x, Tx) < \frac{1}{2} d(x, TTx) = m(x, Tx, T)$. Then by (5) $d(Tx, TTx) < \frac{1}{2} d(x, TTx)$, and after addition of these inequalities we obtain

$$d(x, Tx) + d(Tx, TTx) < \frac{1}{2} d(x, TTx) + \frac{1}{2} d(x, TTx) = d(x, TTx),$$

which is incompatible with the triangle inequality. This contradiction proves (6).

Let $x \in M$ be arbitrary and consider $\{T^n x\}_{n=0}^{\infty}$. Since by (5) and (6) $d(Tx, TTx) \leqslant d(x, Tx)$ for each $x \in M$, we have that

$$(6') \qquad d(T^n x, T^{n+1} x) = d(TT^{n-1} x, TT^n x) \leqslant d(T^{n-1} x, T^n x)$$

for each $n = 1, 2, \ldots$ Thus $\{d(T^n x, T^{n+1} x)\}_{n=0}^{\infty}$ nonincreases and hence has a limit $t \geqslant 0$. Assume (if it is possible) that $t > 0$. Then $\delta(t) > 0$ and, for some k, $d(T^{k-1} x, T^k x) < t + \delta(t)$. Then (4) and (6) imply $d(TT^{k-1} x, TT^k x) = d(T^k x, T^{k+1} x) < t$, which is a contradiction with $d(T^n x, T^{n+1} x) \geqslant t$ for $n = 1, 2, \ldots$ Therefore

$$(7) \qquad \lim_{n\to\infty} d(T^{n-1} x, T^n x) = 0.$$

Now we prove that $\{T^n x\}_{n=0}^{\infty}$ is a Cauchy sequence. Let $\varepsilon>0$ be arbitrary and let $\delta'(\varepsilon)=\min\{\varepsilon, \delta(\varepsilon)\}$. By (7) we may choose n_0 such that

$$d(T^{n-1}x, T^n x)<\delta'(\varepsilon) \quad \text{for} \quad n>n_0. \tag{8}$$

Fix $n>n_0$. It clearly suffices to show that

$$d(T^n x, T^{n+i}x)<\varepsilon, \qquad i=1, 2, \ldots . \tag{9}$$

As for $i=1$ (9) follows from (8) and (6′) immediately, we may proceed by induction on i. Assume that (9) holds for some fixed i. Then by (6′), (8) and (9) we have

$$d(T^{n+i}x, T^{n+i+1}x)<\delta'(\varepsilon),$$

$$d(T^{n-1}x, T^{n+i}x)\leqslant d(T^{n-1}x, T^n x)+d(T^n x, T^{n+i}x)<\varepsilon+\delta'(\varepsilon)$$

and

$$d(T^{n-1}x, TT^{n+i}x)\leqslant d(T^{n-1}x, T^n x)+d(T^n x, T^{n+i}x)+$$
$$+d(T^{n+i}x, T^{n+i+1}x)<\varepsilon+2\delta'(\varepsilon).$$

Hence

$$m(T^{n-1}x, T^{n+i}x, T)=\max\Big\{d(T^{n-1}x, T^{n+i}x),\ d(T^{n-1}x, T^n x),\ d(T^{n+i}x, T^{n+i+1}x),$$
$$\frac{1}{2}[d(T^{n-1}x, TT^{n+i}x)+d(T^{n+i}x, T^n x)]\Big\}<\max\Big\{\varepsilon+\delta'(\varepsilon),\ \delta'(\varepsilon),\ \frac{1}{2}[\varepsilon+2\delta'(\varepsilon)+\varepsilon]\Big\}$$
$$\leqslant\varepsilon+\delta(\varepsilon).$$

Then by (4)

$$d(T^n x, T^{n+i+1}x)=d(TT^{n-1}x, TT^{n+i}x)<\varepsilon.$$

Thus we conclude by induction that (9) is valid for any $n>n_0$ and for all $i=1, 2, \ldots$ Hence $\{T^n x\}_{n=0}^{\infty}$ is the Cauchy sequence and the proof of the first assertion of the Theorem is complete.

Suppose now that M is T-orbitally complete. Then there is some $u\in M$ such that

$$\lim_{n\to\infty} T^n x=u. \tag{10}$$

Suppose now that (4′) is fulfilled and $u\neq Tu$. Since

$$\max\Big\{d(T^n x, u),\ \alpha[d(T^n x, T^{n+1}x)+d(u, Tu)],\ \frac{1}{2}[d(T^n x, Tu)+d(u, T^{n+1}x)]\Big\}$$
$$\leqslant\max\Big\{d(T^n x, u)+d(u, T^{n+1}x)+\alpha\, d(u, Tu)],\ \frac{1}{2}[d(T^n x, u)+$$
$$+d(u, Tu)+d(u, T^{n+1}x)]\Big\}\leqslant d(T^n x, u)+d(u, T^{n+1}x)+qd(u, Tu),$$

where $q=\max\left\{\alpha, \frac{1}{2}\right\}$, we obtain that

$$d(u, Tu)\leqslant d(u, T^{n+1}x)+d(TT^n x, Tu)\leqslant d(u, T^{n+1}x)+[d(T^n x, u)+ + d(u, T^{n+1}x)+qd(u, Tu)]$$

and consequently

$$d(u, Tu)\leqslant\frac{1}{1-q}[2\,d(u, T^{n+1}x)+d(u, T^n x)].$$

Hence, by (10), $d(u, Tu)=0$ and we conclude that $Tu=u$. Therefore, if T in addition satisfies (4′), then T has a fixed point u in M. Clearly, if T is orbitally continuous, then again by (10) $u=Tu$. Since (4) implies that T cannot have more than one fixed point, we have proved (a) and (b).

Letting i tend to infinity in (9) we obtain (c) which provides an estimate for $d(T^n x, u)$.

The proof of the Theorem is complete.

The following example shows that the above theorem is indeed more general than the corresponding theorems of [1], [2], [3], [6], [7] and others.

Example 2. Let $M=[-1, 1]$ be a subset of reals and let T be defined by $T(x)=-1$ for $x>0$ and $T(x)=0$ for $x\leqslant 0$. Then $d(Tx, Ty)=0$ or $d(Tx, Ty)=1$. If $x>0$ and $y\leqslant 0$ then

$$d(Tx, Ty)=1<1+x=d(x, Tx)\leqslant\max\left\{d(x, y),\ d(x, Tx),\ d(y, Ty), \frac{1}{2}[d(x, Ty)+d(y, Tx)]\right\}.$$

Therefore, if in (4) $\varepsilon<1$, then x and y both belong to $[-1, 0]$ or to $(0, 1]$. If $\varepsilon>1$, then $d(Tx, Ty)=1<\varepsilon$. Since there are not x and y in M such that $1=\max\left\{d(x, y),\ d(x, Tx),\ d(y, Ty),\ \frac{1}{2}[d(x, Ty)+d(y, Tx)]\right\}$ we conclude that (4) is satisfied for any $\delta(\varepsilon)>0$ and all $x, y\in M$. Since T is orbitally continuous, we may apply the above theorem. But T is not a generalized contraction since (3) is not satisfied for $y=0$ and $x>0$ which is sufficiently near to zero.

2. Since each mapping which satisfies (2) also satisfy (4) and (4′) (and is continuous), we have as a consequence of our Theorem the following.

Corollary. (*Meir and Keeler*). *Let (X, d) be a complete metric space and T a mapping of X into itself. If* (2) *holds, then T has a unique fixed point u. Moreover, for any $x\in X$, $\lim_{n\to\infty} T^n x=u$.*

Now we may state the following improvement of Theorem 1.

Theorem 2. *Let T be a selfmapping on a T-orbitally complete metric space M. If for some k a mapping $T^k=T(T^{k-1})$ satisfy* (4) *and* (4′), *then T has a unique fixed point u in M and for any $x\in M$ $\lim_{n\to\infty} T^n x=u$.*

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