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A CERTAIN CLASS OF MAPS AND FIXED POINT THEOREMS

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Introduction. In recent years a number of generalizations of Banach contraction principle have appeared, where the authors have discovered new classes of contractive type maps which have fixed points.

In this note we introduce a class of selfmaps on a metric space M which satisfy the following condition

$$d(Tx, Ty) < q \cdot \max\{d(x, y), (d(x, y))^{-1} d(x, Tx)\, d(y, Ty), a(x, y)\, d(x, Ty)\, d(y, Tx)\} \tag{1}$$

for all $x, y \in M$, $x \neq y$, and $q < 1$ or $q = 1$, where $a(x, y)$ is a non-negative real function. These maps may have infinitely many or be without fixed points, even when M is compact. This is shown in § 1, where single-valued maps are investigated. In § 2 multi-valued maps on a generalized metric space which satisfy the condition of the type (1) are discussed.

1. Single-valued maps

Let (M, d) be a metric space and T a selfmapping of M. The following example shows that (1) with $q < 1$ does not imply the existence of a fixed point of T, even if M is compact and $a(x, y) = 0$.

Example. Let $M = [-1, 2] \cup \{8\}$ be a subset of reals with usual metric and let a function T of M into itself be defined by $Tx = -\frac{x}{2}$, if $x \neq 0$ and $x \neq 8$; $T(0) = 8$ and $T(8) = 2$. Then $x, y \in [-1, 2]$ and $x \cdot y \neq 0$ imply $d(Tx, Ty) = \frac{1}{2} d(x, y)$; $y = 0$ and $x \neq 8$ imply

$$d(Tx, To) = 8 + \frac{x}{2} \leqslant 9 = \frac{3}{4} \cdot \frac{\frac{3}{2}|x| \cdot 8}{|x|} = \frac{3}{4}\, \frac{d(x, Tx)\, d(0, To)}{d(x, 0)}$$

and $y=8$ and $x \in M$ imply $d(Tx, T8) \leqslant \frac{1}{2} d(x, 8)$. Therefore, T satisfies (1) with $q=\frac{3}{4}$ and $a(x, y)=0$, but T has not a fixed point. Note that here T is not orbitally continuous.

If T satisfies (1) and has a fixed point, then it need not be unique. For example, the identity mapping satisfies (1) for any $q>0$ and $a(x, y) > (qd(x, y))^{-1}$.

Now we are going to prove some fixed point theorems for maps which satisfy (1). Recall that T is said to be orbitally continuous if $\lim_{i \to \infty} T^{n_i} x = u \in M$ implies $Tu = \lim_{i \to \infty} TT^{n_i} x$ and that M is said to be T-orbitally complete iff every Cauchy sequence of the form $\{T^{n_i} x\}_{i=1}^{\infty}$ converges in M ([1], [2], [3]).

Now we will prove the following result.

Theorem 1. *Let (M, d) be a metric space and $T: M \to M$ a selfmapping of M. If M is T-orbitally complete and T is an orbitally continuous map which satisfies (1), then for each $x \in M$ $\lim_{n \to \infty} T^n x = u_x \in M$ and $Tu_x = u_x$. If in addition $a(x, y) \leqslant (d(x, y))^{-1}$, then T has a unique fixed point.*

Proof. Let $x \in M$ be arbitrary and assume that $Tx \neq x$. Then by (1)

$$d(Tx, T^2x) < q \max \{d(x, Tx), (d(x, Tx))^{-1} d(x, Tx)\, d(Tx, T^2x), 0\} =$$
$$= q \max \{d(x, Tx), d(Tx, T^2x)\}$$

and hence

$$d(Tx, T^2x) < qd(x, Tx).$$

Since $d(Tx, T^2x) = 0 \leqslant qd(x, Tx)$ for the case $Tx = x$, we have

$$d(Tx, T^2x) \leqslant q \cdot d(x, Tx). \tag{2}$$

By the usual procedure from (2) it follows that for any $p \in N$:

$$d(T^n x, T^{n+p} x) \leqslant \frac{q^n}{1-q} d(x\, Tx).$$

Since $q<1$, it follows that $\{x, Tx, T^2, x, \ldots, T^n x, \ldots\}$ is a Cauchy sequence. By orbital completeness of M there exists some $u_x \in M$ such that

$$\lim_{n \to \infty} T^n x = u_x.$$

Since T is orbitally continuous, we have

$$Tu_x = \lim_{n \to \infty} T^{n+1} x = u_x.$$

Therefore, we proved the first part of Theorem.

Let now be $a(x, y) \leqslant \frac{1}{d(x, y)}$ and suppose that $u = Tu$, $v = Tv$ and $v \neq u$. Then

$$d(u, v) = d(Tu, Tv) < q \cdot \max \{d(u, v), 0, (d(u, v))^{-1} d(u, Tv)\, d(v, Tu)\} = qd(u, v)$$

which is a contradiction with $q<1$.

The proof of Theorem is complete.

For maps which satisfy (1) with $q=1$ we have the following result.

Theorem 2. *Let T be an orbitally continuous selfmapping of a metric space M. If T satisfies* (1) *with $q=1$ and for some $x_0 \in M$ the sequence $\{T^n x_0\}_{n=1}^{\infty}$ has a cluster point $u \in M$, then u is a fixed point of T and $\lim_{n\to\infty} T^n x_0 = u$.*

Proof. If $T^n x_0 = T^{n-1} x_0$ for some $n \in N$, then $u = T^{n-1} x_0$ and assertion follows. Assume now that $T^{n-1} x_0 \neq T^n x_0$ for all $n = 1, 2, \ldots,$ and let $\lim_{i\to\infty} T^{n_i} x_0 = u$. Then by (1) (with $q=1$) for $x = T^{n-1} x_0$ and $y = T^n x_0$ we have

$$d(TT^{n-1}x_0, TT^n x_0) < \max\{d(T^{n-1}x_0, T^n x_0),$$

$$(d(T^{n-1}x_0, T^n x_0))^{-1} d(T^{n-1}x_0, T^n x_0)\, d(T^n x_0, T^{n+1}x_0), 0\} =$$

$$= \max\{d(T^{n-1}x_0, T^n x_0), d(T^n x_0, T^{n+1} x_0)\}.$$

Hence

$$d(T^n x_0, T^{n+1} x_0) < d(T^{n-1} x_0, T^n x_0),$$

as $d(T^n x_0, T^{n+1} x_0) < d(T^n x_0, T^{n+1} x_0)$ is impossible. Therefore, the sequence

$$\{d(T^n x_0, T^{n+1} x_0)\}_{n=1}^{\infty}$$

of positive reals is decreasing and hence convergent. Since $\lim_{i\to\infty} T^{n_i} x_0 = u$ and T is orbitally continuous, it follows that $Tu = \lim_{i\to\infty} T^{n_i+1} x_0$, $T^2 u = \lim_{i\to\infty} T^{n_i+2} x_0$ and

$$\lim_{i\to\infty} d(T^{n_i} x_0, T^{n_i+1} x_0) = d(u, Tu), \tag{3}$$

$$\lim_{i\to\infty} d(T^{n_i+1} x_0, T^{n_i+2} x_0) = d(Tu, T^2 u). \tag{4}$$

As $\{d(T^n x_0, T^{n+1} x_0)\}_{n=1}^{\infty}$ is a convergent sequence and

$$\{d(T^{n_i} x_0, T^{n_i+1} x_0)\}_{n=1}^{\infty} \quad \text{and} \quad \{d(T^{n_i+1} x_0, T^{n_i+2} x_0)\}_{n=1}^{\infty}$$

are its subsequences, it follows from (3) and (4) that

$$\lim_{n\to\infty} d(T^n x_0, T^{n+1} x_0) = d(u, Tu) = d(Tu, T^2 u).$$

Therefore, we have

$$d(Tu, T^2 u) = d(u, Tu). \tag{5}$$

Assume that $u \neq Tu$. Then by (1) (with $q=1$) we obtain

$$d(Tu, T^2 u) < d(u, Tu)$$

which contradicts (5). This proves $Tu = u$.

Corollary. *Let M be a compact metric space and $T: M \to M$ an orbitally continuous map of M into M. If T satisfies* (1) *with $q=1$, then for each $x \in M$ $\lim_{n\to\infty} T^n x = u_x$ for some $u_x \in M$ and $Tu_x = u_x$. If in addition $a(x, y) \leqslant (d(x, y))^{-1}$, then T has a unique fixed point.*

Before going into the following theorem, we state the following definitions:

Definition 1. (Kuratowski [6]). Let S be a bounded subset of a metric space M. We denote by $\alpha(S)$, the infimum of all $\varepsilon>0$ such that S admits a finite covering consisting of subsets with diameter less than ε.

Definition 2. (Furi and Vignoli [5]). A continuous mapping $T: M\to M$ of a metric space M into itself is called *densifying* if for every bounded subset S of M such that $\alpha(S)>0$, we have $\alpha(T(S))<\alpha(S)$.

Theorem 3. *Let M be a metric space and $T: M\to M$ be densifying and satisfying* (1) *with $q=1$. If M is T-orbitally complete and for some $x_0\in M$ the sequence of iterates $\{x_0, Tx_0, T^2x_0, \ldots, T^n x_0, \ldots\}$ is bounded, then T has a fixed point.*

Proof. Let $S=\bigcup_{n=0}^{\infty} T^n x_0 (T^0 x_0 = x_0)$. Then $T(S)=\bigcup_{n=1}^{\infty} T^n x_0 \subseteq S$. Also since T is continuous we have $T(\bar{S})\subseteq \overline{T(S)}\subseteq \bar{S}$.

We shall show that $\bar{S}$ is compact. Suppose that $\alpha(S)>0$. Since T is densifying we have $\alpha(T(S))<\alpha(S)$. But $\alpha(S)=\alpha(T(S)\cup\{x_0\})=\alpha(T(S))$. Therefore, $\alpha(S)=0$ and this implies $\alpha(\bar{S})=0$. Hence $\bar{S}$ is totally bounded. By orbitally completeness of M it follows that $\bar{S}$ is compact. Now we may use the Theorem 2.

2. Multi-valued maps

Let (M, d) be a generalized metric space (i. e. a pair (M, d) where M is a set and $d: M\times M\to[0, \infty]$ satisfies all the properties of being a metric for M exept that d may have "infinite values"). A multi-valued function $F: M\to M$ is a point-to-set correspondence. Recall that an orbit of F at the point $x\in M$ is a sequence $\{x_n : x_n\in Fx_{n-1}\}_{n=1}^{\infty}$, where $x_0=x$; F is orbitally upper-semicontinuous if $x_n\to u\in M$ implies $u\in Fu$ whenever $\{x_n\}$ is an orbit of F at some $x\in M$ and that M is F-orbitally complete if every orbit of F at each $x\in M$ which is a Cauchy sequence converges in M ([3], [4]). Let A and B be nonempty subsets of M. Denote

$$D(A, B)=\begin{cases}\inf\{d(a, b): a\in A,\ b\in B\}, & \text{if the infimum exists,}\\ \infty, & \text{otherwise}\end{cases}$$

$$N(\varepsilon, A)=\{x\in M: d(x, a)<\varepsilon \text{ for some } a\in A\},\quad \varepsilon>0,$$

$$H(A, B)=\begin{cases}\inf\{\varepsilon>0: A\subseteq N(\varepsilon, B) \text{ and } B\subseteq N(\varepsilon, A)\}, & \text{if the infimum exists,}\\ \infty, & \text{otherwise}\end{cases}$$

We now state the following result.

Theorem 4. *Let (M, d) be a generalized metric space and let Fx be a closed subset of M for each $x\in M$. If M is F-orbitally complete and F is orbitally upper-semicontinuous and satisfies the following condition*

$$H(Fx, Fy)<q\cdot\max\{d(x, y), (d(x, y))^{-1}D(x, Fx)\,D(y, Fy), \quad a(x, y)\,D(x, Fy)\,D(y, Fx)\} \tag{6}$$

for some $q\in(0, 1)$ and all $x, y\in M$, $x\neq y$ and $a(x, y)\in(0, \infty)$, then F has a fixed point.

P r o o f. Let $t \in (0, 1)$ be arbitrary. We shall define a single-valued function $T: M \to M$ by letting Tx to be a point $y \in Fx$ that satisfies $d(x, y) \leqslant \leqslant q^{-t} D(x, Fx)$.

Let now x be an arbitrary point in M and let us consider the following orbit of F at x:

(7) $$x_0 = x,\ x_1 = Tx_0,\ x_2 = Tx_1,\ \ldots,\ x_n = Tx_{n-1},\ \ldots$$

We may assume that $D(x_n, Fx_n) > 0$ for each $n = 1, 2, \ldots$ since otherwise the assertion of the Theorem follows immediately. Since $x_n \in Fx_{n-1}$ implies $D(x_n, Fx_n) \leqslant \leqslant H(Fx_{n-1}, Fx_n)$ and $D(x_n, Fx_{n-1}) = 0$ we have by (6)

$$D(x_n, Fx_n) \leqslant H(Fx_{n-1}, Fx_n) <$$
$$< q \cdot \max\{d(x_{n-1}, x_n), (d(x_{n-1}, x_n))^{-1} D(x_{n-1}, Fx_{n-1}) D(x_n, Fx_n), 0\}$$
$$\leqslant q \cdot \max\{d(x_{n-1}, x_n), (d(x_{n-1}, x_n))^{-1} d(x_{n-1}, x_n) D(x_n, Fx_n)\}$$
$$= q \cdot \max\{d(x_{n-1}, x_n), D(x_n, Fx_n)\}$$

and hence

(8) $$D(x_n, Fx_n) < q \cdot d(x_{n-1}, x_n).$$

Then by definition of the function T and the sequence (7) it follows from (8) that

$$d(x_n, Tx_n) \leqslant q^{-t} D(x_n, Fx_n) \leqslant q^{-t} q d(x_{n-1}, x_n)$$

and hence

$$d(x_n, x_{n+1}) \leqslant q^{1-t} d(x_{n-1}, x_n).$$

Then, as $q^{1-t} \in (0, 1)$, by routine calculation it may be shown that the orbit (7) of F at x is a Cauchy sequence. As M is F-orbitally complete, there exists some u in M such that

$$\lim_{n \to \infty} x_n = u.$$

Then the orbital upper-semicontinuity of F implies $u \in Fu$, which completes the proof of the Theorem.

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