PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 20 (34) 1976, pp 5–8

CONVERGENT SUBSEQUENCES OF GENERALIZED SEQUENCES OF SETS

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(Received January 21, 1973)

In what follows, we let ω denote the set of all natural numbers and Ω a cardinal number which is cofinal to ω, i. e,, $cf(\Omega)=\omega$.

In this paper we prove that every sequence $(S_i)_{i\in\Omega}$ of type Ω of subsets S_i of natural numbers has a convergent subsequence of type Ω.

Also, we show that for every infinite cardinal c there exists a sequence $(D_u)_{u\in c}$ of type c of subsets D_u of the powerset of c (i. e., $x\in D_u$ implies $x\subseteq c$) such that $(D_u)_{u\in c}$ has no convergent subsequence of any infinite cardinal type.

The above is done by extending the notions of limit superior and limit inferior of denumerable sequences of sets to sequences of sets indexed by arbitrary sets of ordinal numbers.

For related results pertaining to the denumerable case (i. e., $\Omega=c=\omega$) reference is made to [1] and [2].

As usal, we identify every ordinal (and therefore every cardinal number, i.e., an intial ordinal) number with the set of all ordinals less than it (since for ordinals u and v we have $u<v$ if and only if $u\in v$). It then follows that for every set A of ordinal numbers, as well as every family $(A_i)_{i\in q}$ of sets A_i of ordinal numbers, we have

$$(1)\qquad \cup A=\sup A \quad \textit{and} \quad \bigcup_{i\in q} A_i=\sup_{i\in q} A_i$$

We recall that every set A of ordinals is similar to a unique ordinal $\bar{A}$ which is called the *type* of A. Moreover, as expected, if A is a set of ordinals then the sequence $(S_i)_{i\in A}$ is called *of type* $\bar{A}$.

Let A be a set of ordinal numbers. Based on (1), we define *limit superior* $\overline{\lim}_{i\in A} S_i$ of a sequence $(S_i)_{i\in A}$ of sets S_i by:

$$(2)\qquad \overline{\lim_{i\in A}}\, S_i=\{x \mid \cup\{v \mid v\in A \ \textit{and}\ x\in S_v\}=\cup A\}$$

and we define *limit inferior* $\underline{\lim}_{i \in A} S_i$ of $(S_i)_{i \in A}$ by

$$\underline{\lim}_{i \in A} S_i = \{x \mid \cup \{v \mid v \in A \ \ \textit{and} \ \ x \notin S_v\} < \cup A\} \tag{3}$$

From (2) and (3) it follows that $\overline{\lim}_{i \in A} S_i$ as well as $\underline{\lim}_{i \in A} S_i$ always exists (of course, allowing $\varnothing$ as a value) and $\underline{\lim}_{i \in A} S_i \subseteq \overline{\lim}_{i \in A} S_i$. Furthermore, we call $(S_i)_{i \in A}$ *convergent* to $\lim_{i \in A} S_i$ if and only if $\overline{\lim}_{i \in A} S_i = \underline{\lim}_{i \in A} S_i$, in which case

$$\lim_{i \in A} S_i = \overline{\lim}_{i \in A} S_i = \underline{\lim}_{i \in A} S_i \tag{4}$$

In view of (2), (3), (4), we see that $(S_i)_{i \in A}$ is *not convergent* if and only if *for some* m

$$\cup \{v \mid v \in A \ \ \textit{and} \ \ m \in S_v\} = \cup \{v \mid v \in A \ \ \textit{and} \ \ m \notin S_v\} = \cup A \tag{5}$$

Finally, for every set H of ordinal numbers and every ordinal number k we set

$$I_k(H) = \ \textit{the set of the first} \ \ k+1 \ \ \textit{elements of} \ H \tag{6}$$

(when it exists). Thus, $I_0(\{9, \omega+1, 3, \omega+2, \omega\}) = \{3\}$ and $I_2(\{9, \omega+1, 3, \omega+2, \omega\}) = \{3, 9, \omega\}$.

Theorem. *Let* Ω *be a cardinal number which is cofinal to* ω, *i.e.,* $cf(\Omega) = \omega$. *Let* $(S_i)_{i \in \Omega}$ *be a sequence of type* Ω *of subsets* S_i *of* ω. *Then* $(S_i)_{i \in \Omega}$ *has a convergent subsequence of* type Ω, *i.e., there exists a subset* A *of* Ω *such that*

$$\overline{\lim}_{i \in A} S_i = \underline{\lim}_{i \in A} S_i \quad \textit{and} \quad \overline{A} = \Omega \tag{7}$$

Proof. Since $cf(\Omega) = \omega$, there exists a strictly increasing sequence $(r_i)_{i \in \omega}$ of ordinals r_i such that (using notation (1))

$$\bigcup_{i \in \omega} r_i = \Omega \tag{8}$$

For every natural number n, i.e., for every $n \in \omega$, we definie by induction a subset A_n of Ω as follows:

$$A_0 = \{v \mid v \in \Omega \ \ \textit{and} \ \ 0 \in S_v\} \tag{9}$$

provided, $\overline{\{v \mid v \in \Omega \ \ \textit{and} \ \ 0 \in S_v\}} = \Omega$,

otherwise,

$$A_0 = \{v \mid v \in \Omega \ \ \textit{and} \ \ 0 \notin S_v\} \tag{10}$$

and (using notation (6)),

(11) $$A_{n+1}=I_{r_n}(A_n)\cup\{v\mid v\in A_n \text{ and } (n+1)\in S_v\}$$

$$\textit{provided, } \overline{\{v\mid v\in A_n \text{ and } (n+1)\in S_v\}}=\Omega,$$

otherwise,

(12) $$A_{n+1}=I_{r_n}(A_n)\cup\{v\mid v\in A_n \text{ and } (n+1)\notin S_v\}$$

From (6) we see that $I_{r_n}(A_n)\subseteq A_n$ and therefore, in view of (11) and (12), we have $A_{n+1}\subseteq A_n$. Hence,

(13) $$m\leq n \text{ implies } A_n\subseteq A_m\subseteq\Omega \text{ for every } m, n\in\omega$$

Let us observe that if P is a subset of Ω such that $\overline{P}<\Omega$ then $\overline{\Omega-P}=\Omega$. But then from (12) it follows that

(14) $$\overline{A}_n=\Omega \quad \text{for every } n\in\omega$$

Since $(r_i)_{i\in\omega}$ is strictly increasing, from (6), (8), (14) we have

(15) $$\overline{I_{r_n}(A_n)}<\Omega \text{ for every } n\in\omega$$

and

(16) $$\overline{\bigcup_{r\in\omega} I_{r_n}(A_n)}=\Omega$$

Let the subset A of Ω be defined by

(17) $$A=\bigcup_{r\in\omega} I_{r_n}(A_n)$$

Obviously, in view of (6) and (13) we have

(18) $$A\subseteq A_n \text{ for every } n\in\omega$$

We claim that A, as given by (17), is the desired subset of Ω mentioned in (7).

From (16) and (17) it follows that

(19) $$\overline{A}=\Omega \text{ and } \cup A=\Omega$$

Hence, in order to establish (7), in view of (5) it is enough to show that

for no $m\in\omega$ *it is the case that*

(20) $$\cup\{v\mid v\in A \text{ and } m\in S_v\}=\cup\{v\mid v\in A \text{ and } m\notin S_v\}=\cup A=\Omega$$

Let us assume on the contrary that (20) holds for some $m\in\omega$.

We assert that $m\neq 0$. This is because otherwise, by (18) and (20) we would have

(21) $$\cup\{v\mid v\in A_0 \text{ and } 0\in S_v\}=\cup\{v\mid v\in A_0 \text{ and } 0\notin S_v\}=\Omega$$

which is impossible. Indeed, if (9) holds then we see that $\{v\mid v\in A_0 \text{ and } 0\notin S_v)=$ $=\varnothing$ which condradicts (21), and, if (10) holds then $\{v\mid v\in A_0 \text{ and } 0\in S_v\}=\varnothing$ which again contradicts (21).

Hence, $m=n+1$ for some $n\in\omega$. But then from (18) and (20) we have

$$\bigcup\{v \mid v\in A_{n+1} \textit{ and } (n+1)\in S_v\}=\bigcup\{v \mid v\in A_{n+1} \textit{ and } (n+1)\notin S_v\}=\Omega \tag{22}$$

Now, if (11) holds, then by (11), (13), (15) we have:

$$\{v \mid v\in A_{n+1} \textit{ and } (n+1)\in S_v\}\subseteq\{v \mid v\in I_{r_n}(A_n)\}<\Omega$$

which cotradicts (22), and, if (12) holds, then

$$\{v \mid v\in A_{n+1} \textit{ and } (n+1)\notin S_v\}\subseteq\{v \mid v\in I_{r_n}(A_n)\}<\Omega$$

which again contradicts (22).

Thus, our assumption is false and $(S_i)_{i\in A}$ is convergent which implies (7). Hence, the Theorem is proved.

In contradistinction with the above Theorem, we show below that for every infinite cardinal c there exists a sequence $(D_i)_{i\in c}$ with no convergent subsequence of any infinite cardinal type $a\leq c$.

Lemma. *Let c be an infinite cardinal and let $(D_u)_{u\in c}$ be a sequence of subsets D_u of the the powerset of c such that*

$$D_u=\{H \mid H\subseteq c \textit{ and } u\in H\} \textit{ for every } u\in c$$

Then $(D)_{u\in c}$ has no convergent subsequence of any infinite cardinal type a with $a\leq c$.

Proof. Let A be a subset of c such that $\overline{A}=a$. Let $A=B\cup C$ with $B\cap C=\varnothing$ and $\overline{B}=\overline{C}=\overline{A}=a$. Such a decomposition of A exists since $\overline{A}=a$ and a is an infinite cardinal. Let us consider the subset E of c given by

$$E=\{x \mid x\in c \textit{ and } x\notin B\}$$

But then

$$\bigcup\{v \mid v\in A \textit{ and } E\in S_v\}=\bigcup\{v \mid v\in A \textit{ and } E\notin S_v\}=\bigcup A=a$$

which, in view of (20) shows that $(D_u)_{u\in A}$ is not a convergent subsequence of $(D_u)_{u\in C}$. Since A is an arbitrary subset of c with $\overline{A}=a$ (where a is an arbitrary infinite cardinal such that $a\leq c$) we see that the Lemma is established.

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