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ON A PHENOMENON OF OSCILLATING FLOW OF NONHOMOGENEOUS FLUIDS

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The flow phenomena appearing in different cases of flow of nonhomogeneous fluids differ very often essentially from the corresponding phenomena of flow of homogeneous fluids. The phenomena appearing at small forced oscillations of a body in an inviscid nonhomogeneous fluid represent a typical example. It is shown by simple analysis of the type of differential equations describing this flow, that the equations can be of elliptic or hyperbolic type in dependence of it, if the frequency of forced oscillations is greater or less than the Brunt-Väisälä frequency, which depends on a certain way on the law of stratification [1]. In the hyperbolic case the disturbances caused by oscillations of the body propagate along the characteristics through the whole fluid. This phenomenon, which can not occur at flow of a homogeneous fluid at all, was the subject of experimental and theoretical research in papers of a number of authors. It was in an earlier paper [2] shown that the governing equations for this flow can be at a specific law of stratification reduced to the equations with constant coefficients, with what their integration was made easier to a great extent. This law of stratification will be used here in case of oscillations in vertical direction of a horizontal half-infinite plate, in order to show the existence of an interesting and unexpected phenomenon of total absence of any flow in a region of fluid.

The following equation in nondimensional form represents the starting point in investigation of the problem of small forced oscillations of a body in a nonhomogeneous fluid:

$$\frac{\partial^2 \tilde{\psi}}{\partial Z^2}+\frac{\varrho_0'}{\varrho_0}\frac{\partial \tilde{\psi}}{\partial Z}+\left(1+\frac{1}{\beta^2}\frac{\varrho_0'}{\varrho_0}\right)\frac{\partial^2 \tilde{\psi}}{\partial X^2}=0 \tag{1}$$

It can be derived from the Euler equation, the condition of incompressibility of individual particles of fluid and the continuity equation under the following assumptions:

a) perturbations of the density are much less than the density in the state of stable equilibrium, so that the Boussinesq approximation can be used,

b) amplitudes of oscillations are small, so that the linearization of the governing equations can be carried out and

c) the flow is two dimensional.

It is remarkable that in connection with the law of stratification none assumption was made during the deriving of the equation (1), i.e. it can be quite arbitrary. The nondimensional quantities introduced by means of the following scales for length, time, velocity, density and pressure: h_0, $t_0=\sqrt{h_0/g}$, u_0, ρ_0 and $p_0=\rho_0 u_0 \sqrt{g h_0}$ (g-acceleration due to gravity) are marked with capital letters. X and Z are Cartesian coordinates, from which Z points vertically upwards, $Q_0(Z)$ is the density in the state of stable equilibrium, $\beta=\beta_1\sqrt{h_0/g}$, where β_1 represents the frequency of forced oscillations and $\tilde{\psi}$ is the steady-state solution connected with the stream-function ψ in the following way: $\psi=\exp(i\beta T)\tilde{\psi}$ (T-time).

By the transformation of the coordinate Z:

$$\zeta=\int_0^Z \frac{dZ}{Q_0(Z)},$$

under the assumption that the cited integral converges, the equation (1) reduces to the form:

$$\frac{\partial^2\tilde{\psi}}{\partial\zeta^2}+\left(Q_0^2+\frac{Q_0 Q_0'}{\beta^2}\right)\frac{\partial^2\tilde{\psi}}{\partial X^2}=0. \tag{2}$$

This equation becomes an equation with constant coefficients if:

$$Q_0^2+\frac{Q_0 Q_0'}{\beta^2}=\pm a^2,$$

where a is an arbitrary constant, whereby it will be obviously of the elliptic type in case of upper sign, and of the hyperbolic type in case of lower sign. With $Q_0(0)=1$ we obtain:

$$Q_0^2(Z)=(1\mp a^2)\exp(-2\beta^2 Z)\pm a^2. \tag{3}$$

Such a law of stratification is very specific one and artificial too, because it contains the frequence β, from which it naturally does not depend. In an eventual experiment, however, it could be realized relatively easily. Besides, the flow phenomena appearing in a nonhomogeneous fluid probably does not depend qualitatively on the law of stratification, thus we will adopt it here. It must be $a<1$ in the elliptic case, because the stability condition: $Q_0'<0$ would not be fulfilled otherwise. In the hyperbolic case the density becomes zero on a height H. Consequently, the fluid must be bounded in the direction of the axis Z on a height $H_0<H$. The scale h_0 for length can be chosen to be $H_0=1$. The coordinate ζ will be in the hyperbolic case, to which we will restrict ourselves here:

$$\frac{1}{a\beta^2}\operatorname{arctg}\frac{a(1-Q_0)}{a^2+Q_0}. \tag{4}$$

We will consider the flow following the Fig. 1. The half-infinite plate $z=0$, $x>0$ oscillates in vertical direction with the frequency β_1, while the half — infinite plate $z=0$, $x<0$ and the infinite plate $z=h_0$ stay fixed. The boundary conditions will be:

$$Z=0\,(\zeta=0): \frac{\partial\tilde{\psi}}{\partial X}=\begin{cases}-1, & X>0\\ 0, & X<0\end{cases}$$

$Z=1\,(\zeta=\zeta_0)$ and $Z\to-\infty\,(\zeta\to-\zeta_1)$:

$$\frac{\partial\tilde{\psi}}{\partial X}=0, \text{ for all } X.$$

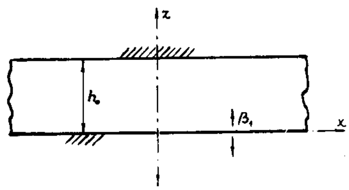

Fig. 1

ζ_0 and ζ_1 can be easily calculated from (4). In order to solve the equation (2) at the specific law of stratification (3) and with the cited boundary conditions, we will use the complex Fourier integral transform [3]:

$$\tilde{\tilde{\psi}}=\int_{-\infty}^{\infty}\tilde{\psi}\,e^{ikX}\,dX, \quad k=\sigma+i\tau, \quad \tau>0.$$

Under the assumptions that: $e^{ikX}\,\tilde{\psi}\Big|_{-\infty}^{\infty}=0$ and $e^{ikX}\dfrac{\partial\tilde{\psi}}{\partial X}\Big|_{-\infty}^{\infty}=0$, for which one can later show, using the solution that will be obtained, that they are satisfied, it will be obtained from the equation (2) in the hyperbolic case:

$$\tilde{\tilde{\psi}}''+a^2k^2\tilde{\tilde{\psi}}=0$$

with the boundary conditions:

$$\tilde{\tilde{\psi}}(0)=\frac{1}{k^2}, \quad \tilde{\tilde{\psi}}(\zeta_0)=0 \text{ and } \tilde{\tilde{\psi}}(-\zeta_1)=0.$$

The solution of this equation in the region $\zeta>0\,(Z>0)$ is:

$$\tilde{\tilde{\psi}}=\frac{\sin ak\,(\zeta_0-\zeta)}{k^2\sin ak\,\zeta_0}.$$

Hence, by means of the inverse formula for the complex Fourier transform we obtain:

$$(5) \qquad 2\pi\tilde{\psi}=\int_{i\tau-\infty}^{i\tau+\infty}\frac{\sin ak\,(\zeta_0-\zeta)}{k^2\sin ak\,\zeta_0}\,e^{-ikX}\,dk.$$

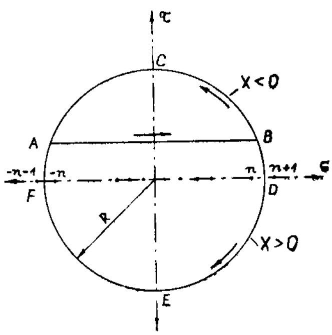

Fig. 2

The integrand has a double pole for $k=0$ and simple poles for $k=\pm n\pi/a\zeta_0$, $n=1, 2, 3, \ldots$, We will choose the contour of integration according to the Fig. 2.

It can be shown on the usual way that the integral (5) tends to zero for $X<\infty$ when $R\to\infty$ on the part BCE of the contour. It is the same case on the part DEF for $X>0$. On the parts BD and

FA the integrand tends to zero for $X>0$ when $R\to\infty$, thus the integral tends to zero too, because the arc remains finite. It can be shown too, that the integral tends to zero in the neighbourhood of the points D and F, because the contour intersects the axis σ between the poles. So we will obtain $\tilde{\psi}\equiv 0$ for $X<0$! The calculation of the residua in the poles yields:

$$\tilde{\psi}=-X\left(1-\frac{\zeta}{\zeta_0}\right)-2a\zeta_0\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2\pi^2}\sin n\pi\left(1-\frac{\zeta}{\zeta_0}\right)\sin\frac{n\pi}{a\zeta_0}X,\quad X>0.$$

Taking into consideration that this expression is a real one, it will be: $\psi=\tilde{\psi}\cos\beta T$, that means that the fluid oscillates everywhere in the same phase with the plate. Making use of the known formula [4]:

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx=\frac{x^2}{4}-\frac{\pi^2}{12},\quad |x|\leqslant\pi,$$

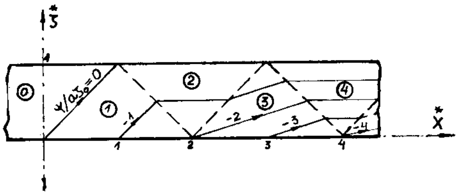

Fig. 3

one can show that $\tilde{\psi}$ can be expressed very simply in the separate parts of the fluid, which are bounded with the characteristic starting from the edge of the plate and whith its reflexions and marked with 0, 1, 2, ... on the Fig. 3.

It will be namely:

$$\tilde{\psi}_{2m}=2ma\zeta_0(\overset{*}{\zeta}-1)\ \text{and}\ \tilde{\psi}_{2m+1}=a\zeta_0[(2m+1)\overset{*}{\zeta}-\overset{*}{X}],\quad m=0,1,2,\ldots$$

where: $\overset{*}{\zeta}=\zeta/\zeta_0$ and $\overset{*}{X}=X/a\zeta_0$, with the velocity components:

$$\tilde{W}_{2m}=0,\quad \tilde{W}_{2m+1}=1\ \text{and}\ \tilde{U}_m=ma/Q_0.$$

The stream-lines in the moment when the plate is moving upwards with the maximal velocity are drawn on the Fig. 3. In accordance with this arrangement of stream-lines we can conclude the following. The region of the total absence of the flow occupies the whole half-plane $X<0$ and the part of the half-plane $X>0$ above the characteristic $\overset{*}{\zeta}=\overset{*}{X}$. In the regions, marked with 1, 3, 5, ..., which are leant on the oscillating plate, the vertical velocity is the same in all points and equal to the velocity of the plate, i.e. the fluid oscillates here in vertical direction as if it constitutes a solid body with the plate. In the regions, marked with 2, 4, 6, ..., which are leant on the fixed plate, the vertical velocity is everywhere zero. The flow performs here only in horizontal direction. The horizontal velocity is present in all regions 1, 2, 3, ..., whereby increases in all of them in vertical direction, as well as by crossing from one region into the other in direction of the axis $\overset{*}{X}$. This increase can be so great that the linear theory, applied here, stops to be valid. The increase of the horizontal velocity can be observed on the Fig. 3 after the concentration of the stream-lines in direction of the axis $\overset{*}{X}$, as well. The stream function is everywhere continuous. The velocity components are, however, discontinuous on the characteristics and its reflexions.

In the region $\zeta<0\,(Z<0)$ one can obtain a fully symmetrical arrangement of stream-lines in reference to the axis $\overset{*}{X}$, whereby only ζ_1 is present instead of ζ_0. With regard to the velocity components the difference is only in the behaviour of the horizontal velocity in the separate regions. This velocity decreases to zero when $\zeta \to -\zeta_1$, because Q_0 increases infinitely at the same time.

The corresponding arrangement of stream-lines in the plane (X, Z) has been given on the Fig. 3 for $a = 0, 2$ and $\beta = 1$. The characteristics starting from

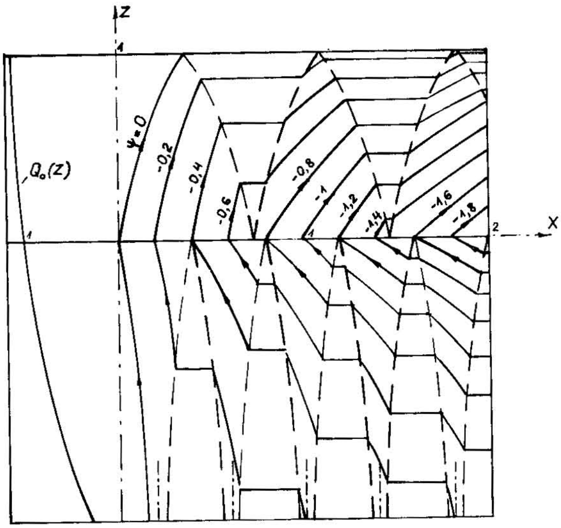

Fig. 4

the edge of the plate, their reflexions from the plate $Z = 1$ and from the infinity $Z = -\infty$ and the stream-lines become curved in this plane. The corresponding law of stratification has been drawn on the left.

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