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ON THE DEPENDENCE OF THE CONTINUOUS SOLUTIONS OF A FUNCTIONAL EQUATION ON AN ARBITRARY FUNCTION

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In this paper we consider the problem of the dependence of continuous solutions of the functional equation of n-th order

(1) $$\varphi[f^n(x)]=g(x, \varphi[f^{n-1}(x)], \ldots, \varphi[f(x)], \varphi(x))$$

on an arbitrary function

This problem was considered by J. Kordylewski and M. Kuczma in [1] for the linear equation of the first order, D. Czaja—Pośpiech and M. Kuczma in [2] for the nonlinear equation of the first order and by B. Choczewski in [3] for the systems of the higher orders (see also in [4] pp. 46—47, 75—77 and 244—254). However the results contained in [1] — [3] do not imply to our Theorem.

Let $I=\langle 0, x_0\rangle$, $0<x_0<\infty$, and assume the following hypotheses about given functions f and g:

(i) $f: I\to R$, is continuous and $0<f(x)<x$, $x\in I\setminus\{0\}$.

(ii) $g: I\times R^n\to R$, continuous;

(2) $g(0, 0, \ldots, 0)=0$;

(3) $$|g(x, y_1, \ldots, y_n)-g(x, \bar{y}_1, \ldots, \bar{y}_n)|\leqslant \sum_{i=1}^{n} s_i|y_i-\bar{y}_i|,$$

$x\in I$, $y_i, \bar{y}_i\in R$, $i=1, 2, \ldots, n$.

We will use the following lemma proved by J. Matkowski in [5]:

L e m m a. *Suppose that* $s_i\geqslant 0$, $i=1, 2, \ldots, n$ *and all the roots of the polynomial*

(4) $$p(z)=z^n-s_1z^{n-1}-\cdots-s_n$$

have the absolute values less than 1. *If* $a_k, b_k\geqslant 0$, $k=1, 2, \ldots$ *fulfil the reccurent inequality*

(5) $$a_{k+n}\leqslant s_1a_{k+n-1}+\cdots+s_na_k+b_k, \qquad k=0, 1, 2, \ldots$$

and $\lim_{k\to\infty} b_k=0$, *then* $\lim_{k\to\infty} a_k=0$.

It is well known, that, under hypotheses (i) and (ii), equation (1) has very many continuous solutions in interval $I\setminus\{0\}$. Namely, the continuous solution of the equation (1) depends on the arbitrary function (compare [4], p. 254). In general, these solutions cannot be extended to the continuous solutions onto the whole interval I. More precisely, $\lim_{x\to 0+}\varphi(x)$ does not exist in general. Nevertheless the following theorem holds:

Theorem. *If hypotheses (i) and (ii) hold and all the roots of the polynomial* (4) *have the absolute values less than* 1, *then every continuous solution* $\varphi: I\setminus\{0\}\to R$ *of the equation* (1) *has the following property*

$$\lim_{x\to 0+}\varphi(x)=0. \tag{6}$$

Proof. Let us put $I_k=\langle f^k(x_0), f^{k-1}(x_0)\rangle$, $k=1, 2, \ldots$. Then we have

$$f^i(I_k)=I_{k+i}, \quad I\setminus\{0\}=\bigcup_{k=1}^{\infty} I_k, \qquad i, k=1, 2, \ldots. \tag{7}$$

Take an arbitrary continuous function φ which fulfils equation (1) in $I\setminus\{0\}$ and put

$$\begin{aligned} a_k &= \sup\{|\varphi(x)|;\ \ x\in I_k\};\\ b_k &= \sup\{|g(x, 0, \ldots, 0)|;\ \ x\in I_k\}. \end{aligned} \tag{8}$$

Let us notice that the relations (6) and $\lim_{k\to\infty} a_k=0$ are equivalent, and therefore it is sufficient to prove the latter. In virtue of (7), (8) and (3) we have

$$\begin{aligned} a_{k+n} &= \sup_{x\in I_{k+n}} |\varphi(x)| = \sup_{x\in I_k} |\varphi[f^n(x)]|\\ &= \sup_{x\in I_k} |g(x, \varphi[f^{n-1}(x)], \ldots, \varphi[f(x)], \varphi(x))\\ &\quad - g(x, 0, \ldots, 0)+g(x, 0, \ldots, 0)|\\ &\leqslant \sum_{i=1}^{n} s_i \sup_{x\in I_k} |\varphi[f^{n-i}(x)]|+b_k\\ &= \sum_{i=1}^{n} s_i \sup_{x\in I_{k+n-i}} |\varphi(x)|+b_k\\ &= \sum_{i=1}^{n} s_i a_{k+n-i}+b_k. \end{aligned}$$

It follows from (2) and from the continuity of the function $g(x, 0, \ldots, 0)$ that $\lim_{k\to\infty} b_k=0$. Since $a_k\geqslant 0$, $b_k\geqslant 0$, $k=1, 2, \ldots$ in virtue of lemma we have

$$\lim_{k\to\infty} a_k=0,$$

which completes the proof.

Remark. If in our theorem we replace the continuity of the function $g(x, 0, \ldots, 0)$ in I by hypothesis that $\lim_{x \to 0} g(x, 0, \ldots, 0) = 0$, then, in general, we loose the continuity of the solution φ in $I \setminus \{0\}$, but the relation (6) remains valid. The proof is analogous.

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