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ON FIXED POINT THEOREMS IN BANACH SPACES

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In [3] the following theorem is proved:

Let (M, d) be a complete metric space and let T be a selfmapping of M into itself such that

$$d(Tx, Ty) \leqslant q \max \left\{ d(x, y),\ d(x, Tx),\ d(y, Ty),\ \frac{1}{2}(d(x, Ty) + d(y, Tx)) \right\} \tag{1}$$

for some $q<1$ and all $x, y \in M$. Then T has a unique fixed point u in M and for each $x \in M$ the sequence $\{T^n x\}$ converges to u.

In subsequent papers [3—7] we studied mappings which satisfy conditions of the type (1).

R. Kannan [10—12] has investigated mappings defined on a bounded closed and convex subset K of a Banach space B and satisfying the following conditions:

$$\| Tx - Ty \| \leqslant \frac{1}{2}(\| x - Tx \| + \| y - Ty \|), \quad x, y \in K \tag{2}$$

and

$$\sup_{z \in D} \| z - Tz \| \leqslant \frac{\delta(D)}{2}, \tag{3}$$

where D is any convex subset of K which is mapped into itself by T ($\delta(D)$ – the diameter of D). In [10] Kannan has proved the following:

(0.1) If B is reflexive and $T: K \to K$ satisfies (2) and (3), then T has a unique fixed point in K;

(0.2) If B is a uniformly convex space and $T: K \to K$ satisfies (2) and (3), then the sequence $\left\{\frac{x_n + Tx_n}{2}\right\}, x_0 \in K$, converges to the fixed point in K.

In [9] and [14] are proved fixed point theorems for continuous mappings which satisfy certain conditions of generalized nonexpansivity. A fixed point for such mappings need not be unique.

In the present note we consider mappings, unnecessarily continuous, which map a bounded closed and convex subset K of a Banach space B into itself and satisfy the following conditions:

$$\| Tx - Ty \| \leqslant \max \left\{ \| x - Tx \|, \quad \| y - Ty \|, \quad \frac{1}{3} (\| x - Ty \| + \| y - Tx \|), \right. \tag{4}$$

$$\left. \frac{1}{3} (\| x - y \| + \| x - Tx \| + \| y - Ty \|) \right\}, \quad x, y \in K$$

and

$$\sup_{z \in D} \| z - Tz \| \leqslant \frac{\delta (D)}{2}, \tag{5}$$

where D is any nonempty closed convex subset of K which is mapped into itself by T. Results here presented are extensions of the cited results of R. Kannan.

1. Before proving theorems, we state one definition and some results. A Banach space B is said to be *uniformly convex* if given $\varepsilon > 0$ there exists $\delta (\varepsilon)$ such that $\| x - y \| \geqslant \varepsilon$ for $\| x \| < 1$ and $\| y \| < 1$ implies that

$$\frac{1}{2} \| x + y \| \leqslant 1 - \delta (\varepsilon).$$

(1.1). Let B be a uniformly convex Banach space and let ε, Q be positive constants. Then there exists a constant q with $0 < q < 1$ such that

$$\| x \| \leqslant Q, \quad \| y \| \leqslant Q, \quad \| x - y \| \geqslant \varepsilon \text{ imply } \| x + y \| \leqslant 2 q \max \{ \| x \|, \| y \| \}.$$

(1.2) Every uniformly convex Banach space is norm-reflexive.

(1.3) A necessary and sufficient condition that a Banach space B be reflexive is that every bounded sequence of nonempty closed convex subsets of B has a nonempty intersection.

Theorem 1. *Let B be a reflexive Banach space, K a nonempty bounded closed and convex subset of B and $T: K \to K$ a mapping satisfying* (4) *and* (5). *Then T has a unique fixed point in K*

Proof. Let $\mathcal{F}$ denote the family of all nonempty closed and convex subsets of K, which T maps into itself. By Zorn's lemma and the result of Smulian [14] it follows that $\mathcal{F}$ has a minimal element, say C.

Define

$$r_x (C) = \sup_{y \in C} \| x - y \|, \quad x \in C,$$

$$r (C) = \inf_{x \in C} r_x (C)$$

and

$$C_c = \{ x \in C : r_x (C) = r (C) \}.$$

The set C_c is nonempty closed and convex.Indeed, if for each $n \in N$ we put $C (x, n) = \left\{ y \in C : \| x - y \| \leqslant r (C) + \frac{1}{n} \right\}$ and $F_n = \bigcap_{x \in C} C (x, n)$, then it follows that $\{ F_n \}$ is a decreasing sequence of nonempty bounded closed and convex sets

Since B is reflexive, it follows (by Theorem (1.3)) that $C_c = \bigcap_{n=1}^{\infty} F_n$ is nonempty closed and convex.

For $x \in C_c$ and $y \in C$ we have

$$\| Tx - Ty \| \leqslant \max \Big\{ \| x - Tx \|, \quad \| y - Ty \|,$$

$$\frac{1}{3}(\| x - Ty \| + \| y - Tx \|), \quad \frac{1}{3}(\| x - y \| + \| x - Tx \| + \| y - Ty \|) \Big\}$$

$$\leqslant \max \left\{ \frac{\delta(C)}{2}, \quad \frac{1}{3}(\| x - Ty \| + \| x - y \| + \| x - Tx \|), \quad \frac{1}{3}(\| x - y \| + \delta(C)) \right\}$$

$$\leqslant \sup_{y \in C} \| x - y \| = r_x(C) = r(C).$$

Therefore, $T(C)$ is contained in a closed spherical ball $\overline{S}[Tx, r(C)]$ and hence $T(C \cap \overline{S}) \subseteq C \cap \overline{S}$ (as $T(C) \subseteq C$). Then by the minimality of C, we get $C \subseteq \overline{S}$. Hence $\| Tx - y \| \leqslant r(C)$ for each $y \in C$. So,

(6) $$\sup_{y \in C} \| Tx - y \| \leqslant r(C).$$

Hence

$$r_{Tx}(C) = \sup_{y \in C} \| Tx - y \| \leqslant r(C),$$

which implies that

$$r_{Tx}(C) = r(C), \quad \text{i.e.,} \quad Tx \in C_c.$$

Therefore, we proved that

(7) $$T(C_c) \subseteq C_c.$$

We claim that, if C contains more than one element, then C_c is a proper subset of C. Suppose not, i.e., $C_c = C$. Then for $x, y \in C$

$$r_x(C) = r_y(C) = r(C).$$

Therefore, $\sup_{z \in C} \| x - z \| = \sup_{z \in C} \| y - z \|$ for $x, y \in C$. This clearly implies that $\sup_{z \in C} \| x - z \| = r(C)$, a constant for all $x \in C$. Hence $\delta(C) = \sup_{x, z \in C} \| x - z \| = r(C)$

This implies that for $x \in C = C_c$ (and $Tx \in C_c$)

(8) $$\sup_{z \in C} \| Tx - z \| = \delta(C).$$

Now by condition (4) we have for $x, y \in C = C_c$

$$\| Tx = Ty \| \leqslant \max \Big\{ \| x - Tx \|, \| y - Ty \|, \quad \frac{1}{3}(\| x - Ty \| + \| y - Tx \|),$$

$$\frac{1}{3}(\| x - y \| + \| x - Tx \| + \| y - Ty \|) \Big\}$$

$$\leqslant \max \left\{ \frac{\delta(C)}{2}, \quad \frac{2}{3}\delta(C), \quad \frac{1}{3}(\delta(C) + \delta(C)) \right\} \leqslant \frac{2}{3}\delta(C).$$

Proceeding in the same manner as in (6) we now get $\sup_{y\in C}\| Tx-y\|\leqslant\frac{2}{3}\delta(C)$, which contradicts (8) because C contains more than one element.

Thus we conclude that if C contains more than one element, then C_c is a proper subset of C. But then, in view of (7), it contradicts the minimality of C. Therefore, C contains only one element, say u. Since T maps C into itself, u is a fixed point of T.

The uniquennes of the fixed point u follows by (4). Indeed, if $v=Tv$ then

$$\| u-v\|=\| Tu-Tv\|\leqslant\max\left\{0,\ \frac{1}{3}(\| u-v\|+\| v-\sigma\|),\ \frac{1}{3}(\| u-v\|)\right\}=\frac{2}{3}\| u-v\|$$

and hence $\| u-v\|=0$. Therefore, $v=u$.

The proof of the Theorem is complete.

Now we prove a Theorem similar to that of Krasnoselskii [13] and [8], but for different kind of mappings.

Theorem 2. *Let K be a nonempty bounded closed and convex subset of a uniformly convex Banach space B and $T:K\to K$ a mapping satisfying*

$$(4')\qquad \| Tx-Ty\|\leqslant\max\left\{\frac{1}{2}(\| x-Tx\|+\| y-Ty\|),\ \frac{1}{3}(\| x-Ty\|+\| y-Tx\|),\ \frac{1}{3}(\| x-y\|+\| x-Tx\|+\| y-Ty\|).\right\}$$

and (5). Then the sequence $\{x_n\}_{n=0}^{\infty}$, where $x_{n+1}=\frac{x_n+Tx_n}{2}$, converges to the fixed point of T in K, where x_0 is any arbitrary point of K.

Proof. Since every uniformly convex Banach space is reflexive, and (4') implies (4), by Theorem 1 T has a unique fixed point u in K. Then we have

$$\| Tx_n-u\|=\| Tx_n-Tu\|\leqslant\max\left\{\frac{1}{2}(\| x_n-Tx_n\|+0),\ \frac{1}{3}(\| x_n-u\|+\| u-Tx_n\|),\ \frac{1}{3}(\| x_n-u\|+\| x_n-Tx_n\|)\right\}$$

$$\leqslant\max\left\{\frac{1}{2}(\| x_n-u\|+\| u-Tx_n\|),\ \frac{1}{3}(\| x_n-u\|+\| x_n-u\|+\| u-Tx_n\|)\right\}.$$

As $\| Tx_n-u\|\leqslant\frac{1}{2}(\| x_n-u\|+\| u-Tx_n\|)$ implies $\| Tx_n-u\|\leqslant\| x_n-u\|$, and $\| Tx_n-u\|\leqslant\frac{1}{3}(2\| x_n-u\|+\| u-Tx_n\|)$ implies $\| Tx_n-u\|\leqslant\| x_n-u\|$, it follows

$$(9)\qquad \| Tx_n-u\|\leqslant\| x_n-u\|.$$

We now consider the sequence $\{\|x_n - Tx_n\|\}$. Assume that

$$\|x_n - Tx_n\| \geqslant \varepsilon > 0 \quad \text{for all} \quad n \in N.$$

Then

$$\|(x_n - u) - (Tx_n - u)\| = \|x_n - Tx_n\| \geqslant \varepsilon, \quad n \in N.$$

Hence, as B is uniformly convex and $x_n, Tx_n \in K$,

$$2\|x_{n+1} - u\| = 2\left\|\frac{x_n + Tx_n}{2} - u\right\| = \|(x_n - u) + (Tx_n - u)\|$$

$$\leqslant 2q \max\{\|x_n - u\|, \ \|Tx_n - u\|\}, \quad n > n_0, \quad 0 < q < 1.$$

Then by (9) we obtain

$$\|x_{n+1} - u\| \leqslant q \cdot \|x_n - u\|, \quad n > n_0, \quad 0 < q < 1.$$

Hence $\lim\limits_{n\to\infty} \|x_n - u\| = 0$. Thus $\lim\limits_{n\to\infty} x_n = u$ when $\inf\limits_{n\in N} \{\|x_n - Tx_n\|\} > 0$.

Suppose now that

$$\inf\{\|x_n - Tx_n\|\} = 0.$$

If $\|x_n - Tx_n\| = 0$ for some n, then x_n is a fixed point of T and hence $x_n = u$. Then $x_{n+1} = \dfrac{x_n + Tx_n}{2} = \dfrac{u + Tu}{2} = u$ and $\lim\limits_{n\to 8} x_n = u$. Let now $\|x_n - Tx_n\| > 0$ for each $n \in N$. Then there exists a subsequence $\{x_{n_i}\}$ of the sequence $\{x_n\}$ such that

$$\lim_{i\to\infty} \|x_{n_i} - Tx_{n_i}\| = 0. \tag{10}$$

By (4) we have

$$\|Tx_{n_i} - Tx_{n_j}\| \leqslant \max\left\{\frac{1}{2}(\|x_{n_i} - Tx_{n_i}\| + \|x_{n_j} - Tx_{n_j}\|),\right.$$

$$\frac{1}{3}(\|x_{n_i} - Tx_{n_j}\| + \|x_{n_j} - Tx_{n_i}\|)$$

$$\left.\frac{1}{3}(\|x_{n_i} - x_{n_j}\| + \|x_{n_i} - Tx_{n_i}\| + \|x_{n_j} - Tx_{n_j}\|)\right\}$$

$$\leqslant \max\left\{\frac{1}{2}(\|x_{n_i} - Tx_{n_i}\| + \|x_{n_j} - Tx_{n_j}\|), \ \frac{1}{3}(\|x_{n_i} - Tx_{n_i}\| + \|Tx_{n_i} - Tx_{n_j}\| + \right.$$

$$+ \|x_{n_j} - Tx_{n_j}\| + \|Tx_{n_j} - Tx_{n_i}\|), \ \frac{1}{3}(\|x_{n_i} - Tx_{n_i}\| + \|Tx_{n_i} - Tx_{n_j}\| + \|Tx_{n_j} - x_{n_j}\| +$$

$$\left. + \|x_{n_i} - Tx_{n_i}\| + \|x_{n_j} - Tx_{n_j}\|)\right\} \leqslant \frac{2}{3}(\|x_{n_i} - Tx_{n_i}\| + \|x_{n_j} - Tx_{n_j}\| + \|Tx_{n_i} - Tx_{n_j}\|).$$

Hence

$$\|Tx_{n_i} - Tx_{n_j}\| \leqslant 2(\|x_{n_i} - Tx_{n_i}\| + \|x_{n_j} - Tx_{n_j}\|).$$

So by (10) $\{Tx_{n_i}\}_{i=1}^{\infty}$ is a Cauchy sequence and hence it converges, say, to v. Then by (10)

$$\lim_{i\to\infty} x_{n_i} = \lim_{i\to\infty} Tx_{n_i} = v. \tag{11}$$

Now

$$\|v-Tv\|\leqslant\|v-x_{n_i}\|+\|x_{n_i}-Tx_{n_i}\|+\|Tx_{n_i}-Tv\|\leqslant\|v-x_{n_i}\|+\|x_{n_i}-Tx_{n_i}\|+$$

$$+\max\left\{\frac{1}{2}(\|x_{n_i}-Tx_{n_i}\|+\|v-Tv\|),\ \frac{1}{3}(\|x_{n_i}-Tv\|+\|v-Tx_{n_i}\|),\right.$$

$$\left.\frac{1}{3}(\|x_{n_i}-v\|+\|x_{n_i}-Tx_{n_i}\|+\|v-Tv\|)\right\}\leqslant\|v-x_{n_i}\|+\|x_{n_i}-Tx_{n_i}\|+$$

$$+\frac{1}{2}(\|x_{n_i}-v\|+\|v-Tv\|+\|v-Tx_{n_i}\|+\|x_{n_i}-Tx_{n_i}\|).$$

So

$$\|v-Tv\|\leqslant 3(\|v-x_{n_i}\|+\|x_{n_i}-Tx_{n_i}\|)+\|v-Tx_{n_i}\|$$

and hence, by (11), $v=Tv$. Then $v=u$, as T has the unique fixed point u. Using (9) we obtain that

$$\|x_{n+1}-u\|=\left\|\frac{x_n+Tx_n}{2}-\frac{u+u}{2}\right\|\leqslant\frac{1}{2}(\|x_n-u\|+\|Tx_n-u\|)\leqslant\|x_n-u\|,$$

and since $\lim_{n\to\infty} x_{n_i}=v=u$, we have $\lim_{n\to\infty}\|x_u-u\|=0$, i.e., $\lim_{n\to\infty} x_n=u$.

The proof of the Theorem is complete.

Theorem 3. *Let B be a uniformly convex Banach space and let T be a selfmapping of B into itself satisfying (4′) on B and*

$$\sup_{y\in D}\|y-Ty\|\leqslant\frac{\delta(D)}{2}, \tag{5'}$$

where D is any nonempty convex subset of B which is mapped into itself by T. Then T has a fixed point in B if and only if the sequence $\{x_n\}_{n=0}^{\infty}$, $x_{n+1}=\frac{x_n+Tx_n}{2}$, x_0 being an arbitrary point, converges in B.

Proof. If T has a fixed point u in B, then, proceeding in the same manner as in Theorem 2, we obtain $\lim_{n\to\infty} x_n=u$.

Assume now that $\{x_n\}$ converges and let

$$v=\lim_{n\to\infty} x_n.$$

We define an operator F by T as follows

$$Fx=\frac{x+Tx}{2}.$$

Then F maps B into itself and we have

$$\|x_{n+1}-Fv\|=\|Fx_n-Fv\|=\left\|\frac{x_n+Tx_n}{2}-\frac{v+Tv}{2}\right\|\leqslant \tag{12}$$

$$\frac{1}{2}(\|x_n-v\|+\|Tx_n-Tv\|).$$

As $\| z-Tz \| = 2 \| z-Fz \|$ for each $z \in B$, by (4′) we get

$$\| Tx_n - Tv \| \leqslant \max \left\{ \frac{1}{2} (\| x_n - Tx_n \| + \| v - Tv \|), \quad \frac{1}{3} (\| x_n - Tv \| + \| v - Tx_n \|), \right.$$

$$\left. \frac{1}{3} (\| x_n - v \| + \| x_n - Tx_n \| + \| v - Tv \|) \right\}$$

$$\leqslant \max \left\{ \frac{1}{2} (\| x_n - Tx_n \| + \| v - Tv \|), \right.$$

$$\left. \frac{1}{3} (\| x_n - v \| + \| v - Tv \| + \| v - x_n \| + \| x_n - Tx_n \|) \right\}$$

$$\leqslant \frac{1}{2} (2 \| x_n - v \| + \| x_n - Tx_n \| + \| v - Tv \|) = \| x_n - v \| + \| x_n - Fx_n \| + \| v - Fv \|$$

$$= \| x_n - v \| + \| x_n - x_{n+1} \| + \| v - Fv \|.$$

Therefore, we have by (12)

$$\| x_{n+1} - Fv \| \leqslant \frac{1}{2} [\| x_n - v \| + (\| x_n - v \| + \| x_n - x_{n+1} \| + \| v - Fv \|)]$$

$$\leqslant \| x_n - v \| + \frac{1}{2} (\| x_n - x_{n+1} \| + \| v - x_{n+1} \| + \| x_{n+1} - Fv \|).$$

Hence

$$\| x_{n+1} - Fv \| \leqslant 2 \| x_n - v \| + \| x_{n+1} - v \| + \| x_n - x_{n+1} \|.$$

Since $\lim_{n \to \infty} x_n = v$, the above inequality implies $v = Fv$. Thus $v = Fv = \frac{v + Tv}{2}$, which gives $v = Tv$.

The proof of the Theorem is complete.

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