PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 19 (33), 1975, pp. 37—41

SPECTRA OF GRAPHS FORMED BY SOME UNARY OPERATIONS

Dragoš Cvetković

(Received February 14, 1975)

Let the vertices of a graph G be labelled by 1, 2, ..., n. The adjacency matrix of G is the matrix $A = \| a_{ij} \|_1^n$, where $a_{ij} = 1$ if the vertices i and j are adjacent and $a_{ij} = 0$ otherwise. The characteristic polynomial $P_G(\lambda) = \det(\lambda I - A)$ of the matrix A is also called the characteristic polynomial of G. The zeros of $P_G(\lambda)$ are eigenvalues of G and they form the spectrum of G.

In some graph theory problems it is necessary to compute the spectrum of a compound graph, formed by some operations from some simpler graphs. In the literature there exist several relations connecting the spectrum of a compound graph with spectra of graphs from which that graph is derived.

The author has given in [1] many such formulas. [1] contains a survey of related results of other authors. There is also a recent review ot the same topic by A. J. Schwenk [7]. In the present paper we shall compute spectra of graphs formed by some unary operations similar to the formation of a line graph.

The line graph $L(G)$ of an undirected graph G without loops or multiple edges is the graph whose vertices are in a biunique correspondence with the set of edges of the graph G, where two vertices from $L(G)$ are adjacent, if, and only if, the corresponding edges in G have a common vertex.

First we shall mention some known results which will be used later.

Let A and B be the adjacency matrices of graphs G and $L(G)$. The incidence matrix of vertices and edges of the graph G is denoted by R, while D denotes the matrix of vertex degrees, for G. As is known (see, for example, [6]) the following relations hold

$$RR^T = A + D, \tag{1}$$

$$R^T R = B + 2I. \tag{2}$$

For regular graphs of degree r we have $D = rI$ and from (1), (2) the following result from [6] follows.

Theorem 1. *If G is a regular graph of degree r with n vertices and $m\left(=\frac{1}{2}nr\right)$ edges, the following relation holds*

$$P_{L(G)}(\lambda) = (\lambda + 2)^{m-n} P_G(\lambda - r + 2). \tag{3}$$

The spectrum of the total graph of a regular graph was determined in [2].

A graph is said to be semi-regular bipartite, if it has two groups of vertices, where no pair of vertices from the same group is adjacent and where the vertex degrees in each group are mutually equal.

The following theorem was proved in [1].

Theorem 2. *Let G be a semi-regular bipartite graph with n_1 mutually non-adjacent vertices of degree r_1 and n_2 mutually non-adjacent vertices of degree r_2, where $n_1 \geqslant n_2$. Then the relation*

$$P_{L(G)}(\lambda) = (\lambda+2)^{\beta} \sqrt{\left(-\frac{\alpha_1}{\alpha_2}\right)^{n_1-n_2} P_G\left(\sqrt{\alpha_1 \alpha_2}\right) P_G\left(-\sqrt{\alpha_1 \alpha_2}\right)} \tag{4}$$

holds, where $\alpha_i = \lambda - r_i + 2$ $(i = 1, 2)$ and $\beta = n_1 r_1 - n_1 - n_2$.

A lemma from the matrix theory is necessary, too.

Lemma. *If M is a non-singular square matrix, we have*

$$\det \begin{Vmatrix} M & N \\ P & Q \end{Vmatrix} = \det M \det (Q - PM^{-1} N).$$

We proceed now to the determination of the characteristic polynomial for some other unary operations on graphs.

The subdivision graph $S(G)$ of a graph G is the graph obtained by forming a new vertex on every edge of G.

The subdivision graph is a bipartite graph whose adjacency matrix is of the form

$$\begin{Vmatrix} 0 & R^T \\ R & 0 \end{Vmatrix}.$$

By the use of the Lemma and the formula (1) we have for a regular graph G of degree r

$$P_{S(G)}(\lambda) = \det \begin{Vmatrix} \lambda I_m & -R^T \\ -R & \lambda I_n \end{Vmatrix} = \lambda^m \det \left(\lambda I_n - R \frac{I_m}{\lambda} R^T \right)$$

$$= \lambda^{m-n} \det (\lambda^2 I_n - RR^T) = \lambda^{m-n} \det (\lambda^2 I_n - A - r I_n) = \lambda^{m-n} P_G(\lambda^2 - r).$$

We therefore arrive at the following theorem:

Theorem 3. *If G is a regular graph of degree r with n vertices and $m\left(=\frac{1}{2} nr\right)$ edges, then*

$$P_{S(G)}(\lambda) = \lambda^{m-n} P_G(\lambda^2 - r). \tag{5}$$

Let $R(G)$ be the graph obtained from G by associating a new vertex to each edge of G and by joining by edges the ends of each edge of G with the vertex associated to this edge.

The adjacency matrix of $R(G)$ is of the form

$$\begin{Vmatrix} 0_m & R^T \\ R & A \end{Vmatrix}.$$

Theorem 4. *If G is a regular graph of degree r with n vertices and $m\left(=\frac{1}{2}nr\right)$ edges, then*

$$P_{R(G)}(\lambda)=\lambda^{m-n}(\lambda+1)^n P_G\left(\frac{\lambda^2-r}{\lambda+1}\right). \tag{6}$$

Proof.

$$P_{R(G)}(\lambda)=\det\begin{Vmatrix} \lambda I_m & -R^T \\ -R & \lambda I_n - A \end{Vmatrix}=\lambda^m \det\left(\lambda I_n - A - R\frac{I_m}{\lambda}R^T\right)=$$

$$=\lambda^m \det\left(\lambda I_n - A - \frac{1}{\lambda}(A+rI_n)\right)=\lambda^{m-n}(\lambda+1)^n P_G\left(\frac{\lambda^2-r}{\lambda+1}\right).$$

Let further, $Q(G)$ be the graph obtained from G by forming a new vertex on every edge of G and by joining by edges those pairs of these new vertices lying on adjacent edges of G. The adjacency matrix of $Q(G)$ is of the form

$$\begin{Vmatrix} B & R^T \\ R & 0_n \end{Vmatrix}.$$

Arguments similar to those previously used lead to the following theorem.

Theorem 5. *Let G be a graph with n vertices and m edges. Then*

$$P_{Q(G)}(\lambda)=\lambda^{n-m}(\lambda+1)^m P_{L(G)}\left(\frac{\lambda^2-2}{\lambda+1}\right). \tag{7}$$

Proof.

$$P_{Q(G)}(\lambda)=\det\begin{Vmatrix} \lambda I_m - B & -R^T \\ R & \lambda I_m \end{Vmatrix}=\lambda^n \det\left(\lambda I_m - B - R^T\frac{I_n}{\lambda}R\right)$$

$$=\lambda^n \det\left(\lambda I_m - B - \frac{1}{\lambda}(B+2I_m)\right)=\lambda^{n-m}(\lambda+1)^m P_{L(G)}\left(\frac{\lambda^2-2}{\lambda+1}\right).$$

Corollary. If G is a regular graph of degree r, then according to (3) and (7) we have

$$P_{Q(G)}(\lambda)=(\lambda+1)^m P_G\left(\frac{\lambda^2-(r-2)\lambda-r}{\lambda+1}\right). \tag{8}$$

Let us consider now the unary operation $L_2(G)$ defined by $L_2(G)=L(S(G))$. If G is a regular graph, the graph $S(G)$ is semiregular bipartite and, combining Theorems 3 and 2, we get the following theorem.

Theorem 6. *If G is a regular graph of degree r with n vertices and $m\left(=\frac{1}{2}nr\right)$ edges, then*

$$P_{L_2(G)}(\lambda)=[\lambda(\lambda+2)]^{m-n} P_G(\lambda^2-(r-2)\lambda-r). \tag{9}$$

In [3] F. Harary and A. J. Schwenk posed the problem of characterizing the integral graphs, i.e. graphs having the spectrum consisting entirely of integers. The general problem seems to be intractable and only partial results have been given in [3].

Similarly as in [3], it is possible to ask under what conditions, holding for the spectrum of a regular graph G, the graphs $S(G)$, $R(G)$, $Q(G)$ and $L_2(G)$ are integral.

The answers can easily be supplied and we shall mention here only the case with $L_2(G)$. According to Theorem 6, if G is regular of degree r, and has eigenvalues $\mu_1, \dots, \mu_n$, the graph $L_2(G)$ has $m-n$ eigenvalues -2, $m-n$ eigenvalues 0 and the following $2n$ eigenvalues

$$\lambda_{1,2}^{(i)}=\frac{1}{2}\left(r-2\pm\sqrt{r^2+4\mu_i+4}\right), \quad i=1, \dots n.$$

Since $-r\leqslant\mu_i\leqslant r$, for the positive value K of the square root in the last formula we have $r-2\leqslant K\leqslant r+2$. The cases $K=r-1$ and $K=r+1$ cannot occur since $\lambda_{1,2}^{(i)}$ must not be non-integer rational numbers. In this way $L_2(G)$ is integral if and only if the eigenvalues μ_i are equal to $r, -1$ or $-r$. This is the case if and only if G is the union of complete graphs all having a fixed number $s\,(s\geqslant 2)$ of vertices.

Thus, all graphs in the graph sequence $L_2(K_n)$, $n=2, 3, \dots$ are integral. $L_2(K_n)$ has for $n>2$ the eigenvalues r, $r-1$, $0, -1, -2$, with the multiplicities 1, $n-1, \frac{1}{2}(n^2-3n)$, $n-1$ and $\frac{1}{2}(n^2-3n+2)$ respectively.

The graph $L_2(K_4)$ is mentioned in [3] as an integral graph which the authors have not been able to identify as a member of a family of integral graphs. Now we have such a family. $L_2(K_4)$ is the third graph in the sequence $L_2(K_n)$, the first two graphs being the complete graph K_2 on two vertices and the cycle C_6 of length 6.

Finally we want to point out the importance of the investigation of relationships between spectra of compound and of simple (in a certain sense) graphs.

A fundamental problem in spectral graph theory is the following. Suppose that a system of numbers is given. The question is: does a graph whose spectrum is equal to the given system of numbers exist; or, similarly, find all graphs having that system of numbers as their spectrum.

Consider a special case. In [5] D. K. Ray Chaudhury has interpreted in the following way one of his results ([4], with A. J. Hoffman). "*A* $(v, k, \lambda)-$symmetric *BIBD* (balanced incomplete block design) exists if and only if there exists a regular connected graph G on vk vertices with distinct eigenvalues -2, $2k-2$, $k-2\pm\sqrt{k-\lambda}$".

Note that the existence of a symmetric *BIBD* with $v=n^2+n+1$, $k=n+1$, $\lambda=1$ implies the existence of a finite projective plane of order n.

It may be possible that a graph with such a spectrum can be derived from some simpler graphs, using several known relations between spectra of graphs formed by some operations and spectra of graphs on which these operations are made.

REFERENCES

[1] D. M. Cvetković, *Graphs and their spectra*, Univ. Beograd, Publ. Elektrotehn. Fak. Ser. Mat. Fiz. №. 354-№. 356 (1971), 1—50.

[2] D. M. Cvetković, *The spectrum of the total graph of a graph*, Publ. Inst. Math. (Beograd) 16 (30) (1973), 49—52

[3] F. Harary, A. J. Schwenk, *Which graphs have integral spectra*? Graphs and combinatorics (ed. R. Bari, F. Harary), Berlin-Heidelberg-New York, 1974, 45—51.

[4] A. J. Hoffman, D. K. Ray Chaudhuri, *On the line graph of symmetric balanced incomplete block designs*, Trans. Amer. Math. Soc. 116 (1965), 238—252.

[5] D. K. Ray Chaudhuri, *On some connections between graph theory and experimental designs and some recent existence results*, Graph theory and its applications (ed. B. Harris), New York-London 1970, 149—166.

[6] H. Sachs, *Über Teiler, Faktoren und charakteristische Polynome von Graphen*, II, Wiss. Z. Techn. Hochsch. Ilmenau 13 (1967), 405—412.

[7] A. J. Schwenk, *Computing the characteristic polynomial of a graph*, Graphs and combinatorics (ed. R. Bari, F. Harary), Berlin-Heidelberg-New York, 1974, 153—172.