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A NOTE ON REPRODUCTIVE SOLUTIONS

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In the paper [1] S. B. Prešić proved some results about general and reproductive solutions of the equation $\mathfrak{E}$: $r(x)=\top$ at a set $S\neq\varnothing$, where $r:S\to\{\top,\bot\}$, i.e. the unary relation at a given set. Basic result is the theorem about obtaining all reproductive solutions using one general solution. In this paper we reduce the determination problem of general and reproductive solutions to functional equations $AgfA=A$ and $AgA=A$ respectively.

By lemma from [1] general solution of $\mathfrak{E}$ could be defined as a mapping $A:S\to S$ such $(\forall x\in S)(r(x)=\top \Leftrightarrow (\exists t\in S)(x=A(t)))$. A reproductive solution of the same equation is a general solution F such

$$(\forall x\in S)(r(x)=\top \Rightarrow x=F(x)).$$

Denote predicates "A is a general solution" and "F is a reproductive solution" with $\mathcal{S}(A)$ and $\mathcal{R}(F)$ respectively. If $R=\{x\in S \mid r(x)=\top\}\neq\varnothing$, i.e. if exists at least one solution of the given equation then exists at least one mapping S on to R which is, by above definition, a general solution. This statement is immediate consequence of the choice axiom.

So we can ask: Is it possible, using one general solution, to describe in certain way, all general and reproductive solutions?

Following theorems answer this question in one possible direction connected with functional equations.

Theorem 1

$$(\forall A, B:S\to S)(\mathcal{S}(A) \Rightarrow (\mathcal{S}(B) \Leftrightarrow (\exists f, g:S\to S)(B=Ag \wedge A=AgfA)))$$

Proof: Let A be the given general solution, so $\mathcal{S}(A)$, and let $\mathcal{S}(B)$. By above definition:

$$\mathcal{S}(B) \Leftrightarrow (\forall x\in S)(r(x)=\top \Leftrightarrow (\exists t\in S)(x=B(t)))$$

$$\Leftrightarrow (\forall x\in S)(r(x)=\top \Rightarrow (\exists t\in S)(x=B(t))) \wedge (\forall x\in S)(\exists t\in S)(x=B(t)) \Rightarrow r(x)=\top)$$

(by predicate calculus: $((\exists x)A(x) \Rightarrow B) \Leftrightarrow (\forall x)(A(x)\Rightarrow B)$ and

$$(A \Rightarrow (\exists x)B(x)) \Leftrightarrow (\exists x)(A \Rightarrow B(x))$$

where variable x has no free occurences in B and A respectively)

$$\Leftrightarrow (\forall x \in S)(\exists t \in S)(r(x) = \top \Rightarrow x = B(t)) \wedge (\forall x \in S)(\forall t \in S)(x = B(t) \Rightarrow r(x) = \top)$$

(application of the choice axiom by [2])

$$\Leftrightarrow (\exists f : S \to S)(\forall x \in S)(r(x) = \top \Rightarrow x =$$
$$= B(f(x))) \wedge (\forall t \in S)(\forall x \in S)(x = B(t) \Rightarrow r(x) = T)$$

(by predicate calculus $(\forall x)(x = u \Rightarrow A(x)) \Leftrightarrow B(u)$ where term u is free for variable x)

$$\Leftrightarrow (\exists f : S \to S)(\forall x \in S)(r(x) = \top \Rightarrow x = B(f(x))) \wedge (\forall t \in S)(r(B(t)) = \top)$$

(A is the general solution so $r(x) = \top \Leftrightarrow (\exists t \in S)(x = A(t))$)

$$\Leftrightarrow (\exists f : S \to S)(\forall x \in S)((\exists t \in S)(x = A(t)) \Rightarrow x =$$
$$= B(f(x))) \wedge (\forall t \in S)(\exists s \in S)(B(t) = A(s))$$

(application of choice axiom by [2])

$$\Leftrightarrow (\exists f : S \to S)(\forall x \in S)(\forall t \in S)(x = A(t) \Rightarrow$$
$$\Rightarrow x = B(f(x))) \wedge (\exists g : S \to S(\forall t \in S)(B(t) = A(g(t)))$$
$$\Leftrightarrow (\exists f : S \to S)(\forall t \in S)(\forall x \in S)(x = A(t)$$
$$\Rightarrow x = B(f(x))) \wedge (\exists g : S \to S)(B = Ag)$$
$$\Leftrightarrow (\exists f : S \to S)(\forall ; \in S)(A(t) = B(f(A(t)))) \wedge (\exists g : S \to S)(B = Ag)$$
$$\Leftrightarrow (\exists f : S \to S)(BfA) = A \wedge (\exists g : S \to S)(B = Ag)$$
$$\Leftrightarrow (\exists f, g : S \to S)(A = BfA \wedge B = Ag)$$
$$\Leftrightarrow (\exists f, g : S \to S)(B = Ag \wedge A = AgfA)$$

According to the theorem 1 if A is a general solution of $\mathcal{E}$ then all general solutions of $\mathcal{E}$ are given by formula Ag where $g : S \to S$ so that with $f : S \to S$ satisfies the equation $A = AgfA$. Next theorem describes the class of the reproductive solutions.

Theorem 2.

$$(\forall A, F : S \to S)(\mathcal{S}(A) \Rightarrow (\mathcal{R}(F) \Leftrightarrow (\exists g : S \to S)(A = Ag\,A \wedge F = Ag)))$$

Proof: Let A be the given reproductive solution, and let F be the reproductive solution so $\mathcal{S}(A)$ and $\mathcal{R}(F)$. By definition:

$$\mathcal{R}(F) \Leftrightarrow \mathcal{S}(F) \wedge (\forall x \in S)(r(x) = \top \Rightarrow x = F(x))$$

(by the proposition A is the general solution so $r(x) = \top \Leftrightarrow (\exists t \in S)(x = A(t)$.

$$\Leftrightarrow \mathcal{S}(F) \wedge (\forall x \in S)((\exists t \in S)(x = A(t)) \Rightarrow x = F(x))$$
$$\Leftrightarrow \mathcal{S}(F) \wedge (\forall x \in S)(\forall t \in S)(x = A(t) \Rightarrow x = F(x))$$
$$\Leftrightarrow \mathcal{S}(F) \wedge (\forall t \in S)(\forall x \in S)(x = A(t) \Rightarrow x = F(x))$$

$\Leftrightarrow \mathscr{S}(F) \wedge (\forall t \in S)(A(t) = F(A(t))$

$\Leftrightarrow \mathscr{S}(F) \wedge A = FA$

(by theorem 1 $\mathscr{S}(F) \Leftrightarrow (\exists f, g: S \to S)(F = Ag \wedge AgfA = A)$)

$\Leftrightarrow (\exists f, g: S \to S)(F = Ag \wedge AgfA = A) \wedge A = FA$

$\Leftrightarrow (\exists f, g: S \to S)(F = Ag \wedge AgfA = A \wedge A = FA)$

$\Leftrightarrow (\exists f, g: S \to S)(F = Ag \wedge AgfA = A \wedge AgA = A)$

$\Leftrightarrow (\exists g: S \to S)(F = Ag \wedge AgA = A \wedge (\exists f: S \to S)(A = AgfA))$

formula $AgA = A \Rightarrow (\exists f: S \to S)(A = AgfA)$ is true ($f = i_S$ satisfies it) so finally we obtain

$\Leftrightarrow (\exists g: S \to S)(F = Ag \wedge A = AgA) \dashv$

The above theorem states that all reproductive solutions of the equation $\mathscr{E}$ are given by formula Ag where $g: S \to S$ satisfies a stronger condition then in the first case, i.e. the equation $A = AgA$ which is natural because the class of the reproductive solutions is more narrow than the class of general solutions. Following lemma describes the solutions of the equation $AgA = A$, where $f(A)$.

Lemma,

$$AgA = A \Leftrightarrow (\forall x \in S)(r(x) = \top \Rightarrow g(x) \in A^{-1}(x))$$

Proof.

$(\forall x \in S)(r(x) = \top \Rightarrow g(x) \in A^{-1}(x))$

$\Leftrightarrow (\forall x \in S)(r(x) = \top \Rightarrow A(g(x)) = x)$

$\Leftrightarrow (\forall x \in S)((\exists t \in S)(x = A(t)) \Rightarrow A(g(x)) = x)$

$\Leftrightarrow (\forall x \in S)(\forall t \in S)(x = A(t) \Rightarrow A(g(x)) = x)$

$\Leftrightarrow (\forall t \in S)(\forall x \in S)(x = A(t) \Rightarrow A(g(x)) = x)$

$\Leftrightarrow (\forall t \in S)(A(g(A(t))) = A(t))$

$\Leftrightarrow AgA = A \quad \dashv$

The above lemma indicates the close connection between reproductive solutions and the choice functions, i.e. the choice axiom.

REFERENCES

[1] S. B. Prešić, *Ein Satz über reproduktive Lösungen*, Publ. Inst. Math. (Beograd) 14 (28), 1972. qq. 133 — 136

[2] N. Bourbaki, *Théorie des ensembles*, Fasc. résultats, Paris, 1958