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ON THE UNIVALENCE OF SOME ANALYTIC FUNCTIONS

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Let $f(z)=z+\sum_{k=n+1}^{\infty} a_k z^k$ and $g(z)=z+\sum_{k=n+1}^{\infty} b_k z^k$ be analytic and satisfy

$$\left|(1-\delta)^{-1}\left[\frac{f(z)}{\lambda f(z)+(1-\lambda)\,g(z)}-\delta\right]-M\right|<M,$$

$0\leqslant\lambda\leqslant 1$, $M\geqslant 1$ if $\delta\neq 0$, $M>1/2$ if $\delta=0$. Then, we determine the values of R, for which $f(z)$ is univalent and starlike for $|z|<R$ under the assumptions

$$(a)\ \mathrm{Re}\left\{\frac{g(z)}{z}\right\}>\eta \text{ or } (b)\ Re\left\{\frac{zf'(z)}{f(z)}\right\}>\alpha;\ 0\leqslant\eta,\ \alpha<1 \text{ and } 1-m\delta-\delta\geqslant 0,$$

$$m=\frac{M-1}{M}.$$

These results sharpen and generalize the results of G. M. Shah [see: Pacific J. Math. **43** (1972), 239—250].

Let $S=\{f/f(z)=z+\sum_{k=n+1}^{\infty} a_k z^k$, analytic in $|z|<1\}$.

Denote by

$$Sg(M,\lambda,\delta)=\{f/\,|(1-\delta)^{-1}[f(z)\,(\lambda f(z)+(1-\lambda)\,g(z))^{-1}-\delta]-M|<M;$$

$$f,\ g\in S \text{ and } 0\leqslant\delta,\ \lambda<1\}. \text{ Identify } Sg(\infty,\lambda,\delta) \text{ by } Sg(\lambda,\delta).$$

Clearly $f\in Sg(\lambda,\delta)$ satisfy

$$Re\ \{f(z)\,(\lambda f(z)+(1-\lambda)\,g(z))^{-1}\}>\delta;\ 0\leqslant\delta<1. \tag{1.1}$$

If $\lambda=0$ and $g(z)=(zf'(z))^{-1}(f(z))^2$, then f is starlike of order $\beta\,(0\leqslant\beta<1)$. We prove the following:

Theorem 1. *Let* $f \in Sg(M, \lambda, \delta)$ *and* $Re\{z(g(z))^{-1} g'(z)\} > \alpha$.

Then f is univalent and starlike in $|z| < R$ *where R denotes the smallest positive root of the equation* $P(r) = 0$ *where*

(1.2) $$P(r) = [1 - (1 - 2\alpha) r^n] [1 + b r^n] [1 + q r^n] - n(b - q)(1 + r^n) r^2;$$

$b = (m - \lambda q)(1 - \lambda)^{-1}$, $q = -1 + \delta + m\delta \leqslant 0$, $0 \leqslant \delta < 1$ *and with the proper choices of the parameters* $m, \lambda, \delta, \alpha$ *for which* $bR^n \leqslant 1$.

If $bR^n > 1$, *then f is univalent and starlike in* $|z| < R_1$ *where* R_1 *denotes the smallest positive root of the equation* $Q(r) = 0$ *where*

(1.3) $$Q(r) = \frac{[1 - (1 - 2\alpha) r^n}{1 + r^n} + (b - q)\left[n\left(bR_0 + \frac{q}{R_0}\right) - n(b + q) - \left(\frac{1 - b^2 r^{2n}}{r^{n-1}(1 - r^2)}\right) \times \right.$$

$$\left.\left(2a - R_0 + \frac{d^2 - a^2}{R_0}\right)\right]; \quad q \geqslant 0 \text{ and } R_0 = \left[\frac{nqr^{n-1}(1 - r^2) + (1 - q^2 r^{2n})}{nbr^{n-1}(1 - r^2) + (1 - b^2 r^{2n})}\right]^{1/2},$$

and R_1 *is smaller than the smallest positive root of the equation,*

(1.4) $$(1 + qr^n)^2 (1 - br^2) - nqr^{n-1}(1 - r^2)(1 + br^n) = 0.$$

Theorem 2: *Let* $f \in Sg(M, \lambda, \delta)$ *and* $Re\left\{\frac{g(z)}{z}\right\} > \alpha$, $0 \leqslant \alpha \leqslant 1/2$.

Then f is univalent and starlike for $|z| < R$ *where R denotes the smallest positive root of the equation* $E(r) = 0$ *where*

(1.5) $$E(r) = \{[1 + r^n] [1 - (1 - 2\alpha) r^n] - 2n(1 - \alpha) r^n\} [1 + qr^n] [1 + br^n]$$
$$- n[b - q] r^n (1 + r^n) [1 - (1 - 2\alpha) r^n]; \quad q = -1 + m\delta + \delta \leqslant 0 \text{ and}$$

choices of $n, \alpha, \delta, \lambda$ *and m are made such that* $bR^n \leqslant 1$. *If* $bR^n > 1$, *then f is univalent and starlike for* $|z| < R_1$ *where* R_1 *denotes the smallest positive root of the equation* $H(R_0) + 1 + K(r) = 0$ *where*

$$K(r) = -nr^n[(1 + r^n)(1 - (1 - 2\alpha) r^n)]^{-1},$$

$$H(R_0) = Q(r) - \left[\frac{1 - (1 - 2\alpha) r^n}{1 + r^n}\right]$$

and Q(r) is given as in theorem 1.

Remarks: (1.) If $n \geqslant 2$, then $P\left[\left(\frac{1}{p}\right)^{1/n}\right] = b^2[2 - n] + b[4\alpha + 2q + qn - n - 2] + [4q\alpha + qnb - 2\alpha] \leqslant 0$ for $q \leqslant 0$. Hence the smallest positive root of the equation $P(r) = 0$ is smaller than $[1/b]^{1/n}$. This proves that conclusion of the theorem 2 holds true for all choices of m, δ, λ and $-q = 1 - \delta - m\delta \geqslant 0$ provided $n \geqslant 2$.

(2.) If $n = 1 = m = -q$, then $P(r) = [1 - y(\alpha, \lambda)]$ where $y(\alpha, \lambda) = 2\lambda^2 + 2\alpha\lambda - 2\alpha\lambda^2$. Hence $P(r) \leqslant [1 - \max y(\alpha, \lambda)] \leqslant 0$ for $0 \leqslant \alpha < 1$, $0 \leqslant \lambda \leqslant 1/2$, while $P(r) \leqslant 0$, if $\alpha = 0$ and $0 \leqslant \lambda \leqslant 1/\sqrt{2}$. Thus, the conclusion of the first part of the theorem is obtained when $n = 1 = m = -q$ and $0 \leqslant \alpha \leqslant 1$ for $0 \leqslant \lambda \leqslant 1/2$; while if $\alpha = 0$, then for $0 \leqslant \lambda < 1/\sqrt{2}$.

(3) If $\alpha=1/2$ and $b\geqslant 1$, then we find that the root of the equation $E(r)$ is less than $(b)^{-1/n}$, if $n\geqslant 2$. If $\alpha=1/2$ and $b\leqslant 1$, then the smallest positive root of the equation $E(r)$ is trivially less than 1. Hence the conclusion of the first part of theorem 2 is true when $\alpha=1/2$, $n\geqslant 2$ and $q\leqslant 0$.

(4.) If $\alpha=0$ and $n\geqslant 2$, then the same is true in (3) for $q\leqslant 0$.

(5.) If $n=1$, then for $\alpha=0$ we get

$$E(b^{-1/n})=(1/b)^3[(b^2-1)(2+2q-b+q)-4b(b+q)]\equiv(b)^{-3}[T(b)].$$

We find

$$\frac{\partial T(b)}{\partial b}=-2b^2-4b+6qb+1-q^2\leqslant 0.$$

Hence

$$\max_{b\geqslant 1} T(b)=T(1)=-4(1+q)\leqslant 0.$$

Thus the conclusion of the theorem remains true in this case also.

(6.) If $n=1$, $\alpha=1/2$, we get

$$E(1/b)=(1/b)^2[\{(1+b)-1\}(2)(b+q)-(b-q)(b+1)]$$
$$\equiv(1/b)^2 T(b).$$

We find that

$$\frac{\partial T(b)}{\partial b}=2(b+q)+2b-(b-q)(1)-(1+b)$$
$$=b[2+2-1-1]+2q+q-1$$
$$=2b+3q-1\leqslant 0 \text{ if } \lambda\leqslant\frac{1-3q-2m}{3-5q}.$$

Thus $\max_{b\geqslant 1} T(b)=T(1)=4q\leqslant 0$. Hence, for $0\leqslant\lambda\leqslant(1-3q-2m)(3-5q)^{-1}$ and $n=1$, $\alpha=1/2$ and $q\leqslant 0$, the conclusion of theorem 2 remains true.

(7.) All above conclusions include the sharp corollaries of G. M. Shah and also all of his theorems are extended, improved and are made sharp.

2. Proof of Theorems. Since $f\in Sg(M,\lambda,\delta)$ there exists a $w(z)=$ $=a_k z^k+\cdots$ regular in the unit disc and $|w(z)|<1$ for which

(2.1) $$\frac{f(z)}{\lambda f(z)+(1-\lambda)g(z)}=\frac{(1-\delta)(1+w(z))}{1-mw(z)}+\delta.$$

The equation (2.1) leads to

(2.2) $$f(z)=\frac{(1-\lambda)(1-qw(z))g(z)}{(1-\lambda)-(m-q\lambda)w(z)};\ q=-1+m\delta+\delta.$$

(2.2) on logarithmic differentiation yields that

(2.3) $$\frac{zf'(z)}{f(z)}=\frac{zg'(z)}{g(z)}+\frac{[(m+\lambda q)-q(1+\lambda)]zw'(z)}{[1-qw(z)][(1-\lambda)-(m-q\lambda)w(z)]}.$$

Now in the case of theorem 2, we have for some $\Omega(z)$

$$\frac{g(z)}{z}=(1-\alpha)\left[\frac{1+\Omega(z)}{1-\Omega(z)}\right]+\alpha=\frac{1+(1-2\alpha)\Omega(z)}{1-\Omega(z)}.$$

This yields on the logarithmic differentiation that

$$\frac{zg'(z)}{g(z)}=1+\frac{2(1-\alpha)z\Omega'(z)}{[1+(1-2\alpha)\Omega(z)][1-\Omega(z)]}. \tag{2.4}$$

For theorem 1, since $g(z)$ is starlike of order α, we have

$$Re\left\{\frac{zg'(z)}{g(z)}\right\}\geqslant\frac{1-(1-2\alpha)r^n}{1+r^n} \tag{2.5}$$

Thus to conclude our theorems, we need the following lemmas.

L e m m a 1. *With the notations of* (2.4) *we have*:

$$Re\left\{\frac{z\Omega'(z)}{[1+(1-2\alpha)\Omega(z)][1-\Omega(z)]}\right\} \tag{2.6}$$

$$\geqslant\begin{cases}\dfrac{-nr^n}{(1+r^n)[1-(1-2\alpha)r^n]} & \text{if } \alpha\leqslant 1/2.\\ n\left(bR_0+\dfrac{q}{R_0}\right)-n(1+q)-\left(\dfrac{1-r^{2n}}{r^{n-1}(1-r^2)}\right)\left(\dfrac{d^2-a^2}{R_0}-R_0+2a\right)\\ \text{if } q=2\alpha-1\geqslant 0,\ \dfrac{1-qr^{2n}}{1-r^{2n}}=a,\ d=\dfrac{2(1-\alpha)r^n}{1-r^{2n}}.\end{cases}$$

L e m m a 2. *With the notations of* (2.3), *we have*

$$Re\left\{\frac{zw'(z)[}{[1-qw(z)][1-bw(z)]}\right\} \tag{2.7}$$

$$\geqslant\begin{cases}\dfrac{-nr^n}{(1+br^n)(1+qr^n)} & \text{if } q\leqslant 0\\ n\left(bR_0+\dfrac{q}{R_0}\right)-n(b+q)-\left(\dfrac{1-b^2r^{2n}}{r^{n-1}(1-r^2)}\right)\left(\dfrac{d^2-a^2}{R_0}-R_0+2a\right)\\ \text{if } q\geqslant 0 \text{ and } \mathrm{R}_0=\left[\dfrac{nqr^{n-1}(1-r^2)+(1-q^2r^{2n})}{nbr^{n-1}(1-r^2)+(1-b^2r^{2n})}\right]^{1/2}.\end{cases}$$

P r o o f. Since the proof of lemma 1 follows from lemma 2, we will prove lemma 2 only. Let

$$p(z)=\frac{1-qw(z)}{1-bw(z)} \quad \text{and } r^n\leqslant 1/b. \tag{2.8}$$

Then the region of variability of $p(z)$ is given by the following inequality.

(2.9) $$|p(z)-a|\leqslant d$$

(2.10) $$a=\frac{1-qbr^{2n}}{1-b^2r^{2n}},\quad b=\left(\frac{m-q\lambda}{1-\lambda}\right)$$

and

(2.11) $$d=\frac{[(m+\lambda q)-q(1+\lambda)]\,r^n}{(1-\lambda)\,[1-b^2r^{2n}]}.$$

Let us write

$$A(z)=n\,Re\left\{b\,(p(z)-1+q\left(\frac{1}{p(z)}-1\right)\right\}$$

and

$$B(z)=[r^{2n}\,|\,bp(z)-q\,|^2-|\,p(z)-1\,|^2]/r^{n-1}(1-r^2)\,|\,p(z)\,|.$$

Thus, we obtain

(2.12) $$Re\left\{\frac{(b-q)^2\,zw'(z)}{(1-bw(z))\,(1-qw(z))}\right\}\geqslant A(z)-B(z),$$

by using the well known inequality due to Goluzin [1],

$$\left|\frac{zw'(z)-nw(z)}{(1-bw(z))\,(1-qw(z))}\right|\leqslant\frac{(r^{2n}-|\,w(z)\,|^2)}{(1-r^2)\,r^{n-1}\,(1-bw(z))\,(1-qw(z))}.$$

Now, we shall determine the lower bound for $A(z)-B(z)$, which will thereby give the lower bound for (2.12). To this purpose, write $p(z)=a+u+iv$, where u and v are real functions of z and a is given by (2.10). Thus we obtain, with $R_1=|\,p(z)\,|$,

$$A(z)-B(z)\equiv J(u,v)$$

$$=n\left\{b\,(a+u-1)+q\left(\frac{a+u}{R_1^2}-1\right)\right\}-\left(\frac{1-b^2\,r^{2n}}{r^{n-1}\,(1-r^2)}\right)\left(\frac{d^2-u^2-v^2}{R_1}\right).$$

Since $J(u,v)$ is a symmetric function of v, it is enough to consider $v\geqslant 0$. Differentiating $J(u,v)$ with respect to v, we get,

$$\frac{\partial J(u,v)}{\partial v}=\frac{v}{R^4}\,H(u,v)\ \text{where}$$

$$H(u,v)=-2\,(a+u)\,nq+R_1\left(\frac{1-b^2\,r^{2n}}{r^{n-1}\,(1-r^2)}\right)(d^2+2\,R_1^2-u^2-v^2).$$

Clearly, if $-q\geqslant 0$, then $H(u,v)\geqslant 0$. Hence, $J(u,v)$ attains its minimum at $v=0$. This gives us

$$\min_v J(u,v)\equiv J(R)=n\left[b\,(R-1)+q\left(\frac{1}{R}-1\right)\right]-\left(\frac{1-b^2\,r^{2n}}{r^{n-1}\,(1-r^2)}\right)\left(\frac{d^2-u^2}{R}\right).$$

As $q \leqslant 0$ and $d^2 - u^2 \geqslant 0$, it follows that $J(R)$ is an increasing function of R; hence

$$(2.13) \qquad \min_{u,\, v} J(u, v) \equiv \min_R J(R) = J(a-d).$$

$$= n\left[b(a-d-1) + q\left(\frac{1}{a-d} - 1\right)\right].$$

But

$$a - d = \frac{1 - qbr^{2n}}{1 - b^2 r^{2n}} - \frac{(b-q)\, r^n}{(1 - b^2 r^{2n})} = \frac{(1 - br^n)(1 + qr^n)}{(1 - b^2 r^{2n})} = \frac{1 + qr^n}{1 + br^n}.$$

Substituting the above in (2.13) yields that

$$(2.14) \qquad \min_{u,\, v} J(u, v) = n\left[b\left(\frac{1 + qr^n}{1 + br^n} - 1\right) + q\left(\frac{1 + br^n}{1 + qr^n} - 1\right)\right]$$

$$= \frac{n(q-b)\, br^n}{(1 + br^n)} + \frac{qn(n-q)\, r^n}{1 + qr^n} = \frac{n(q-b)\, r^n (b-q)}{(1 + br^n)(1 + qr^n)}.$$

Thus from (2.12) and (2.14) we get

$$(2.15) \qquad Re\left\{\frac{zw'(z)}{[1 - bw(z)]\,[1 - qw(z)]}\right\} \geqslant \frac{-nr^n}{(1 + br^n)(1 + qr^n)}.$$

This proves the first part of lemma 2. In proving the second part we consider again $\frac{\partial J(u, v)}{\partial v} = \frac{v}{R^4} H(u, v)$, where $H(u, v)$ is the same expression which occurred earlier. In this case, we assume $q \geqslant 0$. We notice that $d^2 - u^2 \geqslant 0$. Thus, if we show that $-2(a+u)\, nq + 2 R_1^3 (1 - b^2 r^{2n})/r^{n-1}(1 - r^2) \geqslant 0$, we are through. Thus,

$$-2(a+u)\, nq + 2 R_1^3 (1 - b^2 r^{2n})\, r^{1-n} (1 - r^2)^{-1}$$

$$\geqslant \frac{2(a+u)\left[-nqr^{n-1}(1 - r^2)(1 + br^n) + (1 + gr^n)^2 (1 - br^1)\right]}{r^{n-1}(1 - r^2)(1 + br^n)}$$

$$\equiv \frac{2(a+u)\, T(r)}{r^{n-1}(1 - r^2)(1 + br^n)}.$$

If r_0 denotes the smallest positive root of the equation $T(r) = 0$, then for all $r \leqslant r_1$, $H(u, v) \geqslant 0$. Hence $J(u, v)$ attains its minimum at $v = 0$. Thus, we find that

$$\min_v J(u, v) = n\left[b(R-1) + q\left(\frac{1}{R} - 1\right)\right] - \left(\frac{1 - b^2 r^{2n}}{r^{n-1}(1 - r^2)}\right)\left(\frac{d^2 - u^2}{R}\right)$$

$$= n\left[\left(bR + \frac{q}{R}\right) - (b+q)\right] - \frac{1 - b^2 r^{2n}}{r^{n-1}(1 - r^2)}\left[\frac{d^2 - a^n}{R} - R + 2a\right].$$

But

$$\frac{\partial}{\partial R}(\min_{v} J(u, v)) = n\left[b - \frac{q}{R^2}\right] - \left(\frac{1-b^2 r^{2n}}{r^{n-1}(1-r^2)}\right)\left(-1 - \frac{d^2-a^2}{R^2}\right)$$

$$= \left[nb + \frac{1-b^2 r^{2n}}{r^{n-1}(1-r^2)}\right] - \frac{1}{R^2}\left\{nq - \frac{(1-b^2 r^{2n})(d^2-a^2)}{r^{n-1}(1-r^2)}\right\}$$

$$= \left[\frac{nb\, r^{n-1}(1-r^2)+(1-b^2 r^{2n})}{r^{n-1}(1-r^2)}\right] - \left[\frac{nqr^{n-1}(1-r^2)-(1-b^2 r^{2n})(d^2-u^2)}{r^{n-1}(1-r^2)\, R^2}\right].$$

This implies that minimum of $J(u, v)$ is obtained at $v=0$ and

$$R_0 = \left[\frac{nq\, r^{n-1}(1-r^2)+(1-q^2 r^{2n})}{nb\, r^{n-1}(1-r^2)+(1-b^2 r^{2n})}\right]^{1/2}.$$

Hence:

$$\min_{u,v} J(u, v) = n\left[bR_0 + \frac{q}{R_0} - b - q\right] - \frac{(1-b^2 r^{2n})}{r^{n-1}(1-r^2)}\left[2a - R_0 + \frac{d^2-a^2}{R^2}\right]$$

if $r \leqslant r_0$. This completes the proof of lemma 2.

Now proofs of the theorems are immediate from the above lemmas.

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