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GENERALISATION OF BELL POLYNOMIALS AND RELATED OPERATIONAL FORMULAS

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1. Introduction

Following Bell polynomials [5],

$$H_n(g, h) = (-1)^n e^{-hg} D^n e^{hg}; \quad D \equiv \frac{d}{dx} \tag{1.1}$$

and Singh's generalised Truesdell polynomials [9]

$$T_n^{\alpha}(x, r, p) = x^{-\alpha} e^{px^r} (xD)^n (x^{\alpha} e^{-px^r}), \tag{1.2}$$

Shrivastava [6] recently considered generalised polynomials $G_n(h, g)$, defined as

$$G_n(h, g) = e^{-hg} (xD)^n e^{hg}. \tag{1.3}$$

Singh's generalisation was motivated by the generalisation of Hermite polynomials of Gould-Hopper [4] given by

$$H_n^{(r)}(x, \alpha, p) = (-1)^n x^{-\alpha} e^{px^r} D^n (x^{\alpha} e^{-px^r}). \tag{1.4}$$

(1.2) and (1.4) lead Chandel [2] to define generalised Stirling polynomials $T_n^{(\alpha, k)}(x, r, p)$ as

$$T_n^{(\alpha, k)}(x, r, p) = x^{-\alpha-(k-1)n} e^{px^r} (x^k D)^n (x^{\alpha} e^{-px^r}) \tag{1.5}$$

Chandel's generalisation also took note of the generalised polynomials of Chak [1]

$$G_{n,k}^{(\alpha)}(x) = x^{-\alpha} e^x (x^k D)^n (x^{\alpha} e^{-x}). \tag{1.6}$$

Following above generalisations, we define another generalised polynomials by relation

$$G_n^{(k)}(h, g) = e^{-hg} (x^k D)^n e^{hg}. \tag{1.7}$$

It is interesting to note that all polynomials defined above viz. (1.1), (1.2), (1.3), (1.4), (1.5), and (1.6) are particular cases of (1.7). Also Shrivastva [7] noted that generalised Laguerre polynomials

$$T_{nr}^{(\alpha)}(x, p) = \frac{1}{n!} x^{-\alpha-1-n} e^{px^r} (x^2 D)^n (x^{\alpha+1} e^{-px^r}) \tag{1.8}$$

is particular case of (1.7). Many more special cases of (1.7) can be given. In the present note author proposes to study certain properties of (1.7).

2. Operator θ

Some familiar properties of the operator $x^k D \equiv \theta$, which are useful in the development of this paper are as follows:

(2.1) $\theta^n x^\alpha = (\alpha)^{(k-1, n)} x^{\alpha+(k-1)n}$, where $(\alpha)^{(k,n)} = \alpha(\alpha+k)(\alpha+2k)\ldots(\alpha+nk-k)$,

$$\theta^n (UV) = \sum_{r=0}^{n} \binom{n}{r} (\theta^{n-r} U)(\theta^r V), \tag{2.2}$$

$$e^{t\theta}(U.V) = (e^{t\theta} U)(e^{t\theta} V), \tag{2.3}$$

$$e^{t\theta} f(x) = f\left\{\frac{x}{[1-(k-1)\,tx^{k-1}]^{1/(k-1)}}\right\}, \tag{2.4}$$

$$F(\theta)\{x^\alpha \cdot g(x)\} = x^\alpha F(\alpha x^{k-1} + \theta)\, g(x), \tag{2.5}$$

$$F(\theta)\{e^{hg(x)} \cdot f(x)\} = e^{hg(x)} F[x^k hg'(x) + \theta] f(x), \text{ where } g'(x) = \frac{d}{dx} g(x) \tag{2.6}$$

$$\theta^n f\{z(x)\} = \sum_{m=0}^{n} \frac{(-1)^m}{m!} \frac{d^m}{dz^m} f(z) \sum_{i=0}^{m} (-1)^i \binom{m}{i} z^{m-i} \theta^n z^i. \tag{2.7}$$

3. Explicit expression and generating function:

From (1.7) and (2.7) we obtain an explicit expansion for $G_n^{(k)}(h, g)$ as

$$G_n^{(k)}(h, g) = \sum_{m=0}^{n} \frac{(-1)^m}{m!} h^m \sum_{i=0}^{m} (-1)^i \binom{m}{i} g^{m-i} \theta^n g^i. \tag{3.1}$$

Now

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} e^{-hg} \theta^n e^{hg} = e^{-hg} e^{t\theta} e^{hg},$$

gives us a generating function, by use of (2.4) as

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} G_n^{(k)}(h, g) = \exp\left[h\left\{g\left(\frac{x}{[1-(k-1)\,tx^{k-1}]^{1/(k-1)}}\right) - g(x)\right\}\right], \tag{3.2}$$

which reduces to, in particular for $T_n^{(\alpha,k)}(x, r, p)$, $T_{nr}^{(\alpha)}(x, p)$, $T_n^{\alpha}(x, r, p)$ and $H_n^{(r)}(x, \alpha, p)$ respectively as

$$(3.3)\qquad \sum_{n=0}^{\infty} \frac{t^n}{n!} T_n^{(\alpha\ k)}(x, r, p) = [1-(k-1)t]^{\frac{-\alpha}{k-1}} \cdot \exp[px^r\{1-(1-(k-1)t)^{-\frac{r}{k-1}}\}],$$

$$(3.4)\qquad \sum_{n=0}^{\infty} t^n T_{nr}^{(\alpha)}(x, p) = (1-t)^{-\alpha-1} \exp[px^r\{1-(1-t)^{-r}\}],$$

$$(3.5)\qquad \sum_{n=0}^{\infty} \frac{t^n}{n!} T_n^{\alpha}(x, r, p) = \exp[\alpha t + px^r(1-e^{rt})],$$

$$(3.6)\qquad \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n^{(r)}(x, \alpha, p) = \left(1-\frac{t}{x}\right)^{\alpha} \cdot \exp\left[px^r\left\{1-\left(1-\frac{t}{x}\right)^r\right\}\right].$$

By simple manipulations, from (3.2) we also obtain following addition formulas for $G_n^{(k)}(h, g)$ as

$$(3.7)\qquad G_n^{(k)}(h, f+g) = \sum_{m=0}^{n} \binom{n}{m} G_{n-m}^{(k)}(h, g)\, G_m^{(k)}(h, f),$$

$$(3.8)\qquad G_n^{(k)}(h+l, g) = \sum_{m=0}^{n} \binom{n}{m} G_{n-m}^{(k)}(h, g)\, G_m^{(k)}(l, g),$$

and

$$(3.9)\qquad G_n^{(k)}(a, hg+lf) = \sum_{m=0}^{n} \binom{n}{m} G_{n-m}^{(k)}(ah, g)\, G_m^{(k)}(al, f),$$

where a, h and l are parameters and f and g are functions of x. As a special case (3.8) we obtain

$$(3.10)\qquad G_n^{(k)}(2h, g) = \sum_{m=0}^{n} \binom{n}{m} G_{n-m}^{(k)}(h, g)\, G_m^{(k)}(h, g).$$

4. Operator Φ and some relations:

From (2.6) and (1.7) we obtain

$$(4.1)\qquad G_n^{(k)}(h, g) = (x^k hg' + \theta)^n \cdot 1.$$

Putting $\Phi = x^k hg' + \theta$, we have

$$(4.2)\qquad G_n^{(k)}(h, g) = \Phi^n \cdot 1.$$

This operator Φ generalises analoguous operators those given by Gould--Hopper [4]

$$(4.3)\qquad \mathfrak{C} = D + \frac{\alpha}{x} - p\, r x^{r-1},$$

by Singh [9]

$$(4.4)\qquad \mathfrak{C} = x D + \alpha - pr\, x^r,$$

Shrivastava [6]

(4.5) $$\mathfrak{D} = x\,D + x\,hg'$$

and Chandel [2]

(4.6) $$\mathfrak{C} = x^k\,D + \alpha x^{k-1} - pr\,x^{k+r-1}.$$

From (2.2) and (1.7) we obtain

(4.7) $$\theta^m\,G_n^{(k)}(h,\ g) = \sum_{i=0}^{m} \binom{m}{i} G_{m-i}^{(k)}(-h,\ g)\,G_{n+i}^{(k)}(k,\ g),$$

which for $m = 1$, gives

(4.8) $$\theta\,G_n^{(k)}(h,\ g) = -x^k\,hg'\,G_n^{(k)}(h,\ g) + G_{n+1}^{(k)}(h,\ g)$$

or

(4.9) $$\Phi\,G_n^{(k)}(h,\ g) = G_{n+1}^{(k)}(h,\ g).$$

From (4.9) repeated operations of Φ yields

(4.10) $$\Phi^m\,G_n^{(k)}(h,\ g) = G_{n+m}^{(k)}(h,\ g).$$

From (4.2) and (4.10), we observe that

(4.11) $$\Phi^{n+m}\cdot 1 = \Phi^n\,G_m^{(k)}(h,\ g) = \Phi^m\,G_n^{(k)}(h,\ g) = G_{n+m}^{(k)}(h,\ g).$$

It is easily proved by induction that

(4.12) $$\Phi^n(U\cdot V) = \sum_{i=0}^{n} \binom{n}{i} (\theta^i\,U)\cdot(\Phi^{n-i}\,V).$$

From above, we immediately obtain

(4.13) $$\Phi^n\cdot f = \sum_{i=0}^{n} \binom{n}{i} G_{n-i}^{(k)}(h,\ g)\,\theta^i f.$$

When $f = 1$, (4.13) yields (4.2) and when $f = G_m^{(k)}(h,\ g)$, we get

(4.14) $$G_{n+m}^{(k)}(h,\ g) = \sum_{i=0}^{n} \binom{n}{i} G_{n-i}^{(k)}(h,\ g)\,\theta^i\,G_m^{(k)}(h,\ g).$$

This is a generalisation of many similar results for

$$T_{nr}^{\alpha}(x,\ p) \quad \text{and} \quad H_n^r(x,\ \alpha,\ p)$$

discussed by Chatterjea [3] and Singh [8] respectively.

(4.7) and (4.10) suggests

(4.15) $$\theta^m = \sum_{i=0}^{m} \binom{m}{i} G_{m-i}^{(k)}(-h,\ g)\,\Phi^i.$$

Which is easily verified by method of induction. (4.15) happens to be the inverse relation to (4.13). Here it is noticed that relation analoguous to (4.15) for (1.3) should have been

$$\delta^m=\sum_{i=0}^{m}\binom{m}{i}G_{m-i}(-h, g)\mathcal{D}^i, \tag{4.16}$$

and that analoguous to (4.7) should have been

$$\delta^m G_n(h, g)=\sum_{i=0}^{m}\binom{m}{i}G_{m-i}(-h, g)\,G_{n+i}(h, g), \tag{4.17}$$

where $\delta\equiv xD$ and $\mathcal{D}$ is (4.5), instead of those given in [6].

Further we observe from relation (4.13) that

$$e^{t\Phi}\cdot f(x)=\sum_{n=0}^{\infty}\frac{t^n}{n!}G_n^{(k)}(h, g)\cdot e^{t\theta}f(x),$$

which with the help of (2.4) and (3.2) yields

$$e^{t\Phi}\cdot f(x)=\exp\left[h\left\{g\left(\frac{x}{[1-(k-1)\,t\,x^{k-1}]^{1/(k-1)}}\right)-\right.\right.$$
$$\left.\left.-g(x)\right\}\right]\times f\left\{\frac{x}{[1-(k-1)\,t\,x^{k-1}]^{1/(k-1)}}\right\}. \tag{4.18}$$

In particular, when $f(x)=1$,

$$e^{t\Phi}\cdot 1=\exp\left[h\left\{g\left(\frac{x}{[1-(k-1)\,t\,x^{k-1}]^{1/(k-1)}}\right)-g(x)\right\}\right]$$

which by (4.2) implies that

$$e^{t\Phi}\cdot 1=\sum_{n=0}^{\infty}\frac{t^n}{n!}G_n^{(k)}(h, g)=\exp\left[h\left\{g\left(\frac{x}{[1-(k-1)\,t\,x^{k-1}]^{1/(k-1)}}\right)-g(x)\right\}\right] \tag{4.19}$$

and when $f(x)=G_m^{(k)}(h, g)$, we obtain the following generating function

$$e^{t\Phi}G_m^{(k)}(h, g)=\sum_{n=0}^{\infty}\frac{t^n}{n!}G_{n+m}^{(k)}(h, g) \tag{4.20}$$
$$=\exp\left[h\left\{g\left(\frac{x}{[1-(k-1)\,t\,x^{k-1}]^{1/(k-1)}}\right)-g(x)\right\}\right]\cdot$$
$$\cdot G_m^{(k)}\left[h,\, g\left(\frac{x}{[1-(k-1)\,t\,x^{k-1}]^{1/(k-1)}}\right)\right].$$

(4.20) also gives generalisations for many similar generating functions specially for $H_n^{(r)}(x, \alpha, p)$, $T_{nr}^{(\alpha)}(x, p)$, $T_n^{(\alpha)}(x, r, p)$, and $T_n^{(\alpha, k)}(x, r, p)$ etc.

From (4.15) we obtain an addition relation

$$\sum_{i=0}^{m}\binom{m}{i} G_{m-i}^{(k)}(-h, g)\, G_i^{(k)}(h, g)=0 \tag{4.21}$$

5. Some more operational formulas:

Consider,

$$\begin{aligned}\theta^n(e^{hg}f) &= \theta^{n-1} e^{hg}(x^k hg' + x^k D)\cdot f \\ &= \theta^{n-1}\{x^{-r} e^{hg}(x^{k+r} hg' + x^{k+r} D)\cdot f\} \\ &= \theta^{n-2}\{x^{-2r} e^{hg}(x^{k+r} hg' - r\, x^{k+r-1} + x^{k+r} D)\times \\ &\qquad (x^{k+r} hg' + x^{k+r} D)\cdot f\}\end{aligned}$$

which by repeated operation of θ gives

$$e^{-hg}\theta^n(e^{hg}f) = x^{-nr}\prod_{i=1}^{n}\{x^{k+r} hg' - (n-i)\, r\, x^{k+r-1} + x^{k+r} D\}\cdot f \tag{5.1}$$

where product on the right is in operative sense and r is an arbitrary parameter.

Also

$$\begin{aligned}\theta^n(e^{hg}f) &= \sum_{i=0}^{n}\binom{n}{i}\theta^{n-i} e^{hg}\,.\,\theta^i f \\ &= e^{hg}\sum_{i=0}^{n} G_{n-i}^{(k)}(h, g)\,\theta^i f\end{aligned}$$

or

$$e^{-hg}\theta^n(e^{hg}f) = \sum_{i=0}^{n}\binom{n}{i} G_{n-i}^{(k)}(h, g)\cdot\theta^i f. \tag{5.2}$$

Hence from (5.1) and (5.2) we obtain

$$\begin{aligned}&\prod_{i=1}^{n}\{x^{k+r} hg' - (n-i)\, r\, x^{k+r-1} + x^{k+r} D\} f \\ &\qquad = x^{nr}\sum_{i=0}^{n}\binom{n}{i} G_{n-i}^{(k)}(h, g)\cdot\theta^i f.\end{aligned} \tag{5.3}$$

It is interesting to note that (5.3) gives us many new operational formulas for the special functions as particular cases of $G_n^{(k)}(h, g)$.

When $f=1$, (5.3) gives us

$$\prod_{i=1}^{n}\{x^{k+r} hg' - (n-i)\, r\, x^{k+r-1} + x^{k+r} D\}\cdot 1 = x^{nr}\, G_n^{(k)}(h, g). \tag{5.4}$$

In particular, we obtain for $r=0$,

$$(x^k hg' + x^k D)^n\cdot 1 = G_n^{(k)}(h, g) \tag{5.5}$$

which is (4.1).

In case of $H_n^{(s)}(x, \alpha, p)$ with $k=0$, $h=1$ and $g^{(x)}=\alpha \log x - p x^s$, we get

$$(5.6) \qquad \prod_{i=1}^{n} \{x^{r-1}(\alpha - psx^s - (n-i)r) + x^r D\} \cdot 1 = (-1)^n x^{nr} H_n^{(s)}(x, \alpha, p).$$

For $r=0$, (5.6) reduces to the formula given by Gould-Hopper [4].

In case of generalised Laguerre polynomials $T_{ns}^{(\alpha)}(x, p)$, with $k=2$, $h=1$ and $g(x)=(\alpha+1)\log x - p x^s$, we get

$$(5.7) \qquad \prod_{i=1}^{n} [x^{r+1}\{\alpha+1-ps\,x^s-(n-i)r\}+x^{r+2}D] \cdot 1 = n!\, x^{n(r+1)} T_{ns}^{(\alpha)}(x, p).$$

For $r=1$ and $r=0$, (5.7) reduces to the formulas given by Shrivastava [6] as

$$(5.8) \qquad \prod_{i=1}^{n} [x^2\{\alpha+1-ps\,x^s-(n-i)\}+x^3 D] \cdot 1 = n!\, x^{2n} T_{ns}^{(\alpha)}(x, p)$$

and

$$(5.9) \qquad \{x(\alpha+1)-ps\,x^{s+1}+x^2 D\}^n \cdot 1 = n!\, x^n T_{ns}^{(\alpha)}(x, p).$$

In case of generalised Truesdell Polynomials $T_n^{\alpha}(x, s, p)$, with $k=1$. $h=1$ and $g(x)=\alpha \log x - p x^s$, we get

$$(5.10) \qquad \prod_{i=1}^{n} [x^r\{\alpha-ps\,x^s-(n-i)r\}+x^{r+1}D] \cdot 1 = x^{nr} T_n^{\alpha}(x, s, p)$$

which for $r=0$, reduces to the formula given by Singh [9].

And in case of $T_n^{(\alpha,k)}(x, s, p)$, we get

$$(5.11) \qquad \prod_{i=1}^{n} [x^{k+r-1}\{\alpha-ps\,x^s-(n-i)r\}+x^{k+r}D] \cdot 1 = x^{nr+n(k-1)} T_n^{(\alpha,k)}(x, s, p).$$

Many more interesting particular cases of (5.4) can be quoted with the choice of r. It is also interesting to point out that the results (5.6), (5.7) (5.10) and (5.11) are new results and generalise many already known results for the special cases.

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