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HYDROMAGNETIC INSTABILITY IN A ROTATING CHANNEL FLOW

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1. Introduction

Extensive literature exists on the hydromagnetic instability of flows in a non-rotating medium. Stuart [1] and Lock [2] investigated the stability of the flow of an electrically conducting liquid between parallel walls in the presence of a parallel and a transverse magnetic field respectively. They found that for small disturbances, a magnetic field exerts a stabilizing influence on the flow. The stability of Couette flow of a conducting liquid between two parallel plates in the presence of a transverse magnetic field was examined by Kakutani [3], who found that the magnetic field may be destabilizing because of the varying curvature of the basic velocity profile.

In the present paper we have studied the stability of the flow of an electrically conducting liquid permeated by a magnetic field in a channel formed by two vertical parallel plates and placed on a turn table, which is rotated about a vertical axis. The shear flow relative to this rotating frame is induced by either imposing a pressure-gradient along the channel or by moving one plate in its own plane with respect to the other in a Couette flow. The corresponding stability problem for a zonal flow in a non-conducting liquid was recently investigated by Hart [4].

2. Stability of a rotating shear flow with a vertical magnetic field

Consider the Couette flow of an electrically conducting liquid in a vertical channel described above, the liquid being subjected to a uniform vertical magnetic field H_0. The turn table on which the channel is placed is rotated with a uniform angular velocity Ω about a vertical axis which is taken as z-axis. The flow is caused by moving one plate in its own plane relative to the other with velocity U_0 in a horizontal direction which is taken as x-axis. This flow has a velocity distribution $U(y, z)$, along the same direction, the y-axis being perpendicular to the plates. We take the origin in the middle of the channel so that the plates are given by $y=\pm D/2$. When H, the height of the vertical plates is much larger than D, the z-variation in the mean velocity profile $U(y, z)$ is concentrated near the horizontal boundaries. Further, although rotation induces a secondary flow in the y-direction due to Coriolis forces, such a flow will give a small correction to the basic flow $U(y, z)$ if

$$R_0 D/E^2 H \ll 10^3,$$

where R_0 is the Rossby number $U_0/2\Omega D$ and E is the Ekman number $\nu/2\Omega D^2$ (see Hart [4]). We therefore take the basic flow away from the horizontal boundaries as $(U(y), 0, 0)$ for $D/H \ll 1$ and $R_0 D/E^2 H \ll 10^3$, the basic magnetic field being $(0, 0, H_0)$. It can be easily seen that such a magnetic field will not affect the velocity distribution.

We now perturb this basic state by taking the perturbed velocity components as $(U+u, v, w)$ and the perturbed magnetic field components as (h_x, h_y, H_0+h_z). Substituting these in the MHD momentum equations and linearizing, we get the following dimensionless equations

$$R_0\left[\frac{\partial u'}{\partial \tau}+v'\frac{d\bar{U}}{dY}\right]=E\nabla^2 u'+v'+A\frac{\partial h_1'}{\partial Z}, \tag{1}$$

$$R_0\frac{\partial v'}{\partial \tau}=-\frac{\partial p'}{\partial Y}+E\nabla^2 v'-u'+A\frac{\partial h_2'}{\partial Z}, \tag{2}$$

$$R_0\frac{\partial w'}{\partial \tau}=-\frac{\partial p'}{\partial Z}+E\nabla^2 w'+A\frac{\partial h_3'}{\partial Z}, \tag{3}$$

where

$$\bar{U}=\frac{U}{U_0},\quad (u', v', w')=\frac{1}{U_0}(u, v, w),\quad Y=\frac{y}{D},\quad Z=\frac{z}{D},$$

$$\tau=\frac{tU_0}{D},\quad (h_1', h_2', h_3')=\frac{1}{H_0}(h_x, h_y, h_z),\quad p'=\frac{p}{\rho U_0^2} \tag{4}$$

and A is the magnetic parameter $\mu_e H_0^2/8\pi\rho D\Omega U_0$. In writing the above equations, we have assumed $\partial/\partial x=0$ following Benton [5]. Further with $\partial/\partial x=0$, the equation of continuity and the solenoidal relation $\nabla\cdot\vec{H}=0$ enable us to introduce Ψ and Φ such that

$$v'=\frac{\partial\Psi}{\partial Z},\quad w'=-\frac{\partial\Psi}{\partial Y},\quad h_2'=\frac{\partial\Phi}{\partial Z},\quad h_3'=-\frac{\partial\Phi}{\partial Y}. \tag{5}$$

Using (5) in (1) — (3) and eliminating the pressure p', we get

$$[E(D^2-k^2)-R_0\sigma]\,g(Y)-Akm(Y)=\left[1-R_0\frac{d\bar{U}}{dY}\right]kh(Y), \tag{6}$$

$$[E(D^2-k^2)-R_0\sigma](D^2-k^2)h-kg(Y)+Ak(D^2-k^2)\,l(Y)=0, \tag{7}$$

where following the techniques of normal mode analysis we have assumed

$$\Psi = \mathrm{Re}\{ie^{\sigma\tau}e^{ikZ}h(Y)\}; \quad u' = \mathrm{Re}\{e^{\sigma\tau}e^{ikZ}g(Y)\};$$
$$\Phi = \mathrm{Re}\{e^{\sigma\tau}e^{ikZ}l(Y)\}; \quad h_1' = \mathrm{Re}\{ie^{\sigma\tau}e^{ikZ}m(Y)\} \tag{8}$$

with $D \equiv d/dY$ and Re denoting the real part. The disturbances are thus in the form of rolls with their axes along x-axis.

Similarly the magnetic induction equation gives the following components

$$\left[\frac{1}{R_M}(D^2-k^2)-\sigma\right]m(Y) = -k\frac{d\bar{U}}{dY}l(Y) - kg(Y), \tag{9}$$

$$\left[\frac{1}{R_M}(D^2-k^2)-\sigma\right]l(Y) = kh(Y). \tag{10}$$

If the marginal state is stationary, we may put $\sigma = 0$ in the above equations. Elimination of g, l and m in favour of h gives

$$(D^2-k^2)[E(D^2-k^2)^2 + AR_M k^2]^2 h(Y) = \\ = -k^2[(R_0-1)(D^2-k^2)^2 + AR_M^2 k^2]h(Y). \tag{11}$$

Here we have set $d\bar{U}/dY = 1$ for the plane Couette flow velocity field $\bar{U} = Y + (1/2)$ satisfying $\bar{U} = 1$ at $Y = 1/2$ and $\bar{U} = 0$ at $Y = -1/2$. The differential equation (11) being of tenth order requires ten boundary conditions. These are furnished by the four magnetic boundary conditions corresponding to the prescription of $l(Y)$ and $m(Y)$ at $Y = \pm 1/2$ and the six no-slip conditions for velocity corresponding to the vanishing of $h(Y)$, $h'(Y)$ and $g(Y)$ at $Y = \pm 1/2$. For the sake of simplicity we assume the surfaces $Y = \pm 1/2$ to be free so that the shear stress vanishes there. Using (6) – (10), these conditions give

$$h = D^2 h = D^4 h = D^6 h = \cdots = 0 \quad \text{at} \quad Y = \pm\frac{1}{2}. \tag{12}$$

It may be seen that no explicit magnetic boundary conditions are needed for the determination of eigenvalues, although they will be required for determining the components of the magnetic field. It was pointed out by Chandrasekhar [6] that qualitative conclusions about stability are not affected by the assumption of two free surfaces despite its artificial nature. The solution consistent with (12) corresponding to the lowest mode is

$$h(Y) = A_1 \cos \pi Y, \tag{13}$$

A_1 being a constant. This will satisfy (11) provided

$$\frac{1}{E^2} = \frac{[(\pi^2+k^2)^2 + Qk^2]^2}{k^2(R_0-1)(\pi^2+k^2)}, \tag{14}$$

where

$$Q = \frac{AR_M^2}{R_0 P_m}, \quad P_m = \frac{\nu}{\eta} \tag{15}$$

and the last term in (11) is neglected while writing (14) since for mercury, the magnetic Prandtl number $P_m \approx \cdot 16 \times 10^{-6}$. It may be seen that a necessary condition for the marginal stability to exist is $R_0 < 1$. The physical significance of this inequality is that the total vorticity should be negative so that the perturbations may be sustained by coupling between the vorticity field and Corio-

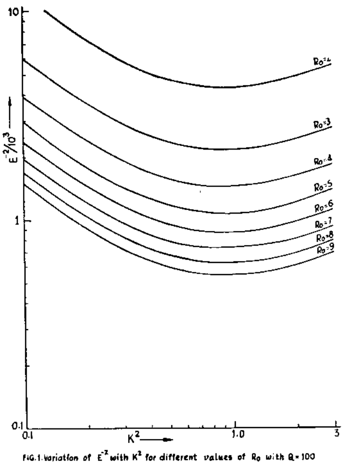

Fig. 1. Variation of E^{-2} with K^2 for different values of R_0 with $Q = 100$

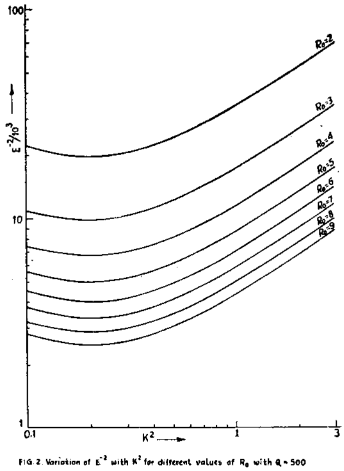

Fig. 2. Variation of E^{-2} with K^2 for different values of R_0 with $Q = 500$

lis force. Variations of E^{-2} with k^2 for several values of the magnetic parameter Q and Rossby number R_0 are shown in Figs. 1 and 2. It may be seen that E^{-2} has a minimum and for a fixed Q, this minimum decreases steadily

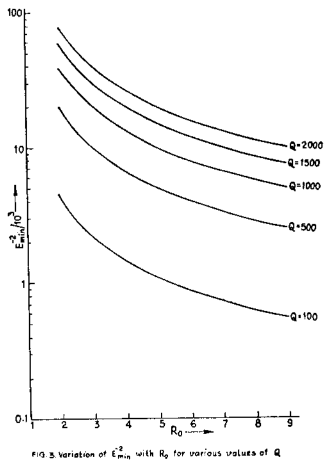

Fig. 3. Variation of $E_{\min}^{-2}$ with R_0 for various values of Q

with increase in R_0. The variation of this minimum (denoted by $E_{\min}^{-2}$) with R_0 for several values of Q is shown in Fig. 3 which shows that $E_{\min}^{-2}$ steadily increases with increase in the magnetic field for a fixed Rossby number. Thus the magnetic field exerts a strong stabilizing influence on the flow.

3. Stability of a rotating pressure flow with a horizontal magnetic field

We next consider the stability of the hydromagnetic flow in the vertical channel (described earlier) rotating about a vertical axis, the motion relative to the rotating frame being caused by a uniform pressure gradient along the channel. The liquid is permeated by a uniform transverse magnetic field along y-axis, so that under the conditions stated earlier viz., $D/H \ll 1$ and $R_0 D/E^2 H \ll 10^3$, the velocity distribution will correspond to Hartmann flow in a non-rotating channel (Cowling [7]). We thus have a basic velocity field $(U(y), 0, 0)$ and a basic magnetic field $(H_x(y), H_0, 0)$ whose stability we shall now investigate. Taking the perturbed velocity field as $(U+u, v, w)$ and the perturbed magnetic field as (H_x+h_x, H_0+h_y, h_z) and assuming the disturbances in the form of rolls as in section 2 (i.e., $\partial/\partial x = 0$), the linearized perturbation equations for momentum are

$$R_0\left(\frac{\partial u'}{\partial \tau} + v'\frac{d\bar{U}}{dY}\right) = E\nabla^2 u' + v' + \frac{\mu_e H_0^2}{8\pi\rho\Omega D U_0}\cdot\left[\frac{\partial h_1'}{\partial Y} + h_2'\frac{dH_x'}{dY}\right], \tag{16}$$

$$R_0\frac{\partial v'}{\partial \tau} = -\frac{\partial p'}{\partial Y} + E\nabla^2 v' - u' + \frac{\mu_e H_0^2}{8\pi\rho\Omega D U_0}\frac{\partial h_2'}{\partial Y}, \tag{17}$$

$$R_0\frac{\partial w'}{\partial \tau} = -\frac{\partial p'}{\partial Z} + E\nabla^2 w' + \frac{\mu_e H_0^2}{8\pi\rho\Omega D U_0}\frac{\partial h_3'}{\partial Y}, \tag{18}$$

where $H_x' = H_x(y)/H_0$ and the dimensionless variables are the same as defined in (4) and U_0 is given by $PM(\cosh M - 1)/\mu_e^2 \sigma H_0^2 \sinh M$, P being the imposed pressure gradient in the channel and M being the Hartmann number $(1/2)\mu_e H_0 d(\sigma/\rho\nu)^{1/2}$. Similarly the components of the magnetic induction equation are

$$\frac{\partial h_1'}{\partial \tau} + v'\frac{dH_x'}{dY} = \frac{\partial u'}{\partial Y} + h_2'\frac{d\bar{U}}{dY} + \frac{1}{R_M}\nabla^2 h_1', \tag{19}$$

$$\frac{\partial h_2'}{\partial \tau} = \frac{\partial v'}{\partial Y} + \frac{1}{R_M}\nabla^2 h_2', \tag{20}$$

$$\frac{\partial h_3'}{\partial \tau} = \frac{\partial w'}{\partial Y} + \frac{1}{R_M}\nabla^2 h_3'. \tag{21}$$

In the above equations, $\bar{U}$ stands for $U(y)/U_0$ where $U(y)$ is the Hartmann velocity given by (Cowling [7])

$$U(y) = \frac{PM}{\mu_e^2 \sigma H_0^2}\left[\frac{\cosh M - \cosh\dfrac{2My}{D}}{\sinh M}\right]. \tag{22}$$

The equation of continuity and $\nabla\cdot\vec{H} = 0$ give rise to two functions ψ and Φ (as in section 2) defined by

$$v' = \frac{\partial \psi}{\partial Z}, \quad w' = -\frac{\partial \psi}{\partial Y}, \quad h_2' = \frac{\partial \Phi}{\partial Z}, \quad h_3' = -\frac{\partial \Phi}{\partial Y}. \tag{23}$$

Use of (23) in (16) gives

$$\left(E\nabla^2 - R_0 \frac{\partial}{\partial \tau}\right)u' + A\left(\frac{\partial h_1'}{\partial Y} + h_2' \frac{dH_x'}{dY}\right) = \left(R_0 \frac{d\bar{U}}{dY} - 1\right)\frac{\partial \psi}{\partial Z}, \tag{24}$$

while elimination of p' from (17) and (18) gives upon using (23)

$$\left(E\nabla^2 - R_0 \frac{\partial}{\partial \tau}\right)\nabla^2 \psi - \frac{\partial u'}{\partial Z} + A\frac{\partial}{\partial Y}(\nabla^2 \Phi) = 0. \tag{25}$$

Further (19) and (23) give

$$\left(\nabla^2 - R_M \frac{\partial}{\partial \tau}\right)h_1' = \left(\frac{dH_x'}{dY}\cdot\frac{\partial \psi}{\partial Z} - \frac{d\bar{U}}{dY}\cdot\frac{\partial \Phi}{\partial Z} - \frac{\partial u'}{\partial Y}\right)R_M, \tag{26}$$

while both (20) and (21) lead to

$$\left(\nabla^2 - R_M \frac{\partial}{\partial \tau}\right)\Phi = -\frac{\partial \psi}{\partial Y}\cdot R_M. \tag{27}$$

To make further progress we shall now assume $R_M \ll 1$ which is a reasonable assumption for flow of conducting liquids like mercury or liquid sodium under laboratory conditions. In this case it is reasonable to assume that both the basic and perturbed induced magnetic field components are small so that equations (26) and (27) will be approximately satisfied. Indroducing

$$\psi = \mathrm{Re}\{ie^{ikZ} h(Y)\}, \tag{28}$$

$$u' = \mathrm{Re}\{e^{ikZ} g(Y)\} \tag{29}$$

for the marginal state (assuming it to be steady) and substituting in (24) and (25), we get after ignoring terms of $O(R_M)$:

$$E(D^2 - k^2)\, g(Y) + \left(R_0 \frac{d\bar{U}}{dY} - 1\right) kh(Y) = 0, \tag{30}$$

$$E(D^2 - k^2)^2 h(Y) - kg(Y) = 0. \tag{31}$$

It may be noticed that although the induced magnetic field does not appear in (30) (31), the basic velocity $\bar{U}(Y)$ is modified by the imposed magnetic field. The boundary conditions are the no-slip conditions at the walls given by

$$h' = h = g = 0 \quad \text{at} \quad Y = \pm\frac{1}{2}. \tag{32}$$

We have solved (30) and (31) by Galerkin's method a succint account of which may be found in Mikhlin [8]. Calculations were performed on an IBM 1620 Digital Computer and the variation of E^{-2} with k^2 is shown in

Fig. 4 for several values of R_0 with $M=4$. This shows that as in the case of a parallel magnetic field, E^{-2} has a minimum $E^{-2}_{\min}$ for fixed M and R_0. Further for a fixed wave number k, E^{-2} decreases with increase in R_0. In Fig. 5,

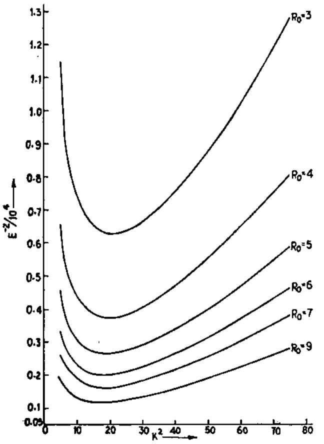

Fig. 4. Variation of E^{-2} with K^2 for different values of R_0 with $M=4$

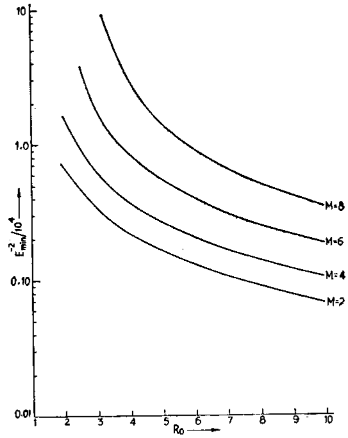

Fig. 5. Variation of $E^{-2}_{\min}$ with R_0 for various values of M

curves of $E^{-2}_{\min}$ versus R_0 are plotted for various values of M which show that the magnetic field exerts a strong stabilizing influence on the flow.

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