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# ON THE RANGE OF A BOOLEAN TRANSFORMATION

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**Abstract.** A Boolean transformation $F: \mathbf{D} \to \mathbf{B}^m$ is defined by a system $F = (f_1, \ldots, f_m)$ of Boolean functions $f_i: \mathbf{B}^n \to \mathbf{B}$ $(i:=1, \ldots, m)$ and a Boolean domain $\mathbf{D} \subseteq \mathbf{B}^n$ (i.e., $\mathbf{D}$ is characterized by a Boolean equation $d(X) = 1$). In this paper we determine the range $F(\mathbf{D})$ of a Boolean transformation and study a few related problems. Previous results of Schröder, Whitehead, Löwenheim, Eggert and the author are obtained as particular cases.

## § 0. Statement of the problem and prerequisites

Let $(B, \cup, \cdot, ', 0, 1)$ be an arbitrary Boolean algebra. A *Boolean function* of $p$ variables is a mapping $f: \mathbf{B}^p \to \mathbf{B}$ which can be obtained from variables and constants by superpositions of the basic operations $\cup, \cdot, '$. A *Boolean equation* is obtained by equating two Boolean functions and in particular by equating a Boolean function to 0 or to 1; it is well known that any (system of) Boolean equation(s) can be brought to this latter form. The set of solutions to a Boolean equation will be called a *Boolean domain*. Let further $n$ and $m$ be two arbitrary but fixed positive integers. A *Boolean transformation* is a mapping $F: \mathbf{D} \to \mathbf{B}^m$, where $\mathbf{D} \subseteq \mathbf{B}^n$ is a Boolean domain defined by a Boolean equation $d(X) = 1$, while $F$ is of the form $F = (f_1, \ldots, f_m)$, where each $f_i$ is a Boolean function $f: \mathbf{B}^n \to \mathbf{B}$ $(i:=1, \ldots, m)$; in other words, $F(X)$ is defined for every $X \in \mathbf{D}$ by $F(X) = (f_1(X), \ldots, f_m(X))$.

In this paper we determine the range $F(\mathbf{D})$ of a Boolean transformation $F$ (theorem 1), thus generalizing the theorem obtained by Whitehead for $m:=1$. As a corollary we deduce a necessary and sufficient condition for $F$ to be surjective, previously obtained in [5] and generalizing a theorem of Whitehead and Löwenheim. A result of Eggart is then obtained as another corollary. Theorem 3 establishes a necessary and sufficient condition in order that $F(\mathbf{D})$ be a given Boolean domain of $\mathbf{B}^m$. Theorems 3 and 4 give necessary and sufficient conditions for $F$ to be a constant and a given constant, respectively. In order to facilitate the understanding of all these results, they are transcribed in § 2 for $m:=1, 2$.

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Before proceeding to the statement and proof of the above announced results, we still need a few prerequisites.

Notations like $X=(x_1, \ldots, x_p)$, $Y=(y_1, \ldots, y_p)$, ... denote vectors with components in $\mathbf{B}$, while $A=(\alpha_1, \ldots, \alpha_p)$, $B=(\beta_1, \ldots, \beta_p)$, ... stand for *elementary vectors*, that is, vectors with components in the subset $\{0, 1\} \subseteq \mathbf{B}$; the number $p$ of components will result from the context. As is well known, Boolean functions $f: \mathbf{B}^p \to \mathbf{B}$ are characterized by the property that they can be written in *canonical disjunctive form*

$$f(X) = \bigcup_A f(A) X^A,$$

where $\bigcup_A$ means that $A=(\alpha_1, \ldots, \alpha_p)$ runs over the $2^p$ elementary vectors, $X^A = x_1^{\alpha_1} \ldots x_p^{\alpha_p}$ and $x^0 = x'$, $x^1 = x$. Quite naturally, notations like

$$F^C(X) = f_1^{\gamma_1}(X) \ldots f_p^{\gamma_p}(X), \quad F^C(A) = f_1^{\gamma_1}(A) \ldots f_p^{\gamma_p}(A)$$

will also be used, where $C=(\gamma_1, \ldots, \gamma_p)$, $f^\gamma(X)$ stands for $[f(X)]^\gamma$, etc. Recall that computation with canonical disjunctive forms is facilitated by the rules

$$\bigcup_A X^A = 1,$$

$$X^A X^B = X^A \text{ if } A=B, \ X^A X^B = 0 \text{ if } A \neq B,$$

$$(\bigcup_A b_A X^A) \cup (\bigcup_A c_A X^A) = \bigcup_A (b_A \cup c_A) X^A,$$

$$(\bigcup_A b_A X^A)(\bigcup_A c_A X^A) = \bigcup_A b_A c_A X^A,$$

$$(\bigcup_A b_A X^A)' = \bigcup_A b_A' X^A;$$

more generally, these rules also apply to *orthonormal systems*, i.e., families of elements $\{a_\lambda\}_{\lambda \in \Lambda}$ such that $\bigcup_{\lambda \in \Lambda} a_\lambda = 1$, $a_\lambda a_\mu = 0$ for $\lambda, \mu \in \Lambda$ and $\lambda \neq \mu$.

Computation is also facilitated by the use of *symmetric difference* $x+y = xy' \cup x'y$, mostly due to the property $x+y=0 \Leftrightarrow x=y$.

The Boolean equation $f(X)=0$ is consistent if and only if $\Pi_A f(A) = 0$. Thus in particular the equation in one unknown $ax \cup bx' = 0$ is consistent if and only if $ab=0$, When this is the case, the set of solutions is the *interval* $[b, a'] = \{x \in \mathbf{B} \mid b \leqslant x \leqslant a'\}$.

The reader is urged to transcribe the above results in *dual* form, interchanging $\cup$ with $\cdot$, 0 with 1, and $\leqslant$ with $\geqslant$. For more details and many other properties of Boolean functions and equations, see [5]

## 1. The range of a Boolean transformation

Throughout this section $n$ and $m$ are two arbitrary but fixed positive integers. Let the letter $X$ (the letters $A$, $B$) denote (elementary) vectors from $\mathbf{B}^n$, while the letter $Y$ (the letters $C$, $D$) will stand for (elementary) vectors from $\mathbf{B}^m$. Let further $\mathbf{D} \subseteq \mathbf{B}^n$ denote the Boolean domain defined by a Boolean equation $d(X)=1$ and

(1) $$F: \mathbf{D} \to \mathbf{B}^m$$

the Boolean transformation defined by the restriction to **D** of a vector $F=(f_1, \ldots, f_m)$ of Boolean functions $f_i : \mathbf{B}^n \to \mathbf{B}$ $(i:=1, \ldots, m)$.

Theorem 1. *The range $F(\mathbf{D})$ of the Boolean transformation* (1) *coincides with the set of solutions $Y$ to the Boolean equation*

$$\bigcup_C [\bigcup_A d(A) F^C(A)] Y^C = 1. \tag{2}$$

Comment. For $m:=1$ we have $F=f : \mathbf{B}^n \to \mathbf{B}$, equation (2) reduces to

$$[\bigcup_A d(A) f(A)] y \cup [\bigcup_A d(A) f'(A)] y' = 1$$

and its set of solutions is the interval

$$[\Pi_A [d'(A) \cup f(A)], \quad \bigcup_A d(A) f(A)]. \tag{3}$$

discovered by Whitehead [7] (see also [5], theorem 2.5).

Proof. The condition $Y \in F(\mathbf{D})$ can be written successively in the following equivalent forms:

$$(\exists X) \; d(X)=1 \;\&\; Y=F(X),$$

$$(\exists X) \bigcup_A d(A) X^A = 1 \;\&\; \prod_{i:=1}^m [y_i f_i(X) \cup y_i' f_i'(X)] = 1,$$

$$(\exists X) \bigcup_A \{d(A) \prod_{i:=1}^m [y_i f_i(A) \cup y_i' f_i'(A)]\} X^A = 1,$$

$$\bigcup_A \{d(A) \prod_{i:=1}^m [y_i f_i(A) \cup y_i' f_i'(A)]\} = 1,$$

$$\bigcup_C [\bigcup_A \{d(A) \prod_{i:=1}^m [\gamma_i f_i(A) \cup \gamma_i' f_i'(A)]\}] Y^C = 1$$

and the latter condition coincides with (2) because

$$\prod_{i:=1}^m [\gamma_i f_i(A) \cup \gamma_i' f_i'(A)] = \prod_{i:=1}^m f_i^{\gamma_i}(A) = F^C(A).$$

Corollary 1. *The Boolean transformation* (1) *is surjective if and only if*

$$\Pi_C \bigcup_A d(A) F^C(A) = 1.$$

Comment. This is theorem 8.2 in [5].

Proof. The equation (2) is identically satisfied if and only if all the coefficients equal 1, i.e.,

$$\bigcup_A d(A) F^C(A) = 1$$

for every $C$.

Corollary 2. *The range $F(\mathbf{B}^n)$ of the Boolean transformation $F : \mathbf{B}^n \to \mathbf{B}^m$ coincides with the set of solutions $Y$ to the Boolean equation*

$$\bigcup_C [\bigcup_A F^C(A)] Y^C = 1. \tag{5}$$

Comment. For $m:=1$ we have $F=f : \mathbf{B}^n \to \mathbf{B}$, equation (5) reduces to

$$[\bigcup_A f(A)] y \cup [\bigcup_A f'(A)] y' = 1$$

and its set of solutions is the range

$$(6)\qquad [\Pi_A f(A),\quad \cup_A f(A)],$$

a classical result due to Schröder [6] (see also [5], theorem 2.4).

P r o o f. Take $d\equiv 1$ in theorem 1.

C o r o l l a r y 3. *The Boolean transformation* $F:\mathbf{B}^n\to\mathbf{B}^m$ *is surjective if and only if*

$$(7)\qquad \Pi_C \cup_A F^C(A)=1.$$

C o m m e n t. This result was first discovered by Whitehead [8] and Löwenheim [4] (see also [5], theorem 8.3).

P r o o f. Take $d\equiv 1$ in corollary 1.

As another application of theorem 1, let us give a shorter proof of the following.

T h e o r e m (Eggart [2]). *Given a Boolean function* $f:\mathbf{B}^n\to\mathbf{B}$ *and two elements* $a, b\in\mathbf{B}$ *with* $a\leqslant b$, *the range of the Boolean transformation*

$$(8)\qquad f:[a, b]^n\to\mathbf{B}$$

*is*

$$(9)\qquad [\Pi_{\Xi\in\{a\ b\}^n}f(\Xi),\quad \cup_{\Xi\in\{a, b\}^n}f(\Xi)].$$

P r o o f. The range $f([a, b]^n)$ of the Boolean transformation (8) is the interval (3) given in theorem 1 for $m:=1$, where $d(X)=1$ is the characteristic equation of the Boolean domain $[a, b]^n$, that is,

$$d(X)=\prod_{j:=1}^n (bx_j\cup a'\,x_j').$$

Therefore, setting $O=(0,\ldots,0)$, $I=(1,\ldots,1)$ and noticing that $a\leqslant b$ implies $a'\cup b=1$, it follows that the range (3) reduces to

$$(10)\qquad \begin{aligned}&[\Pi_A[\cup_{j:=1}^n(b'\,\alpha_j\cup a\,\alpha_j')\cup f(A)],\\ &\cup_A[\prod_{j:=1}^n(b\,\alpha_j\cup a'\,\alpha_j')]f(A)]=\\ &=[[a\cup f(O)][b'\cup f(I)]\,\Pi_{A\neq O,\,I}[a\cup b'\cup f(A)],\\ &a'f(O)\cup bf(I)\cup\cup_{A\neq O,I}\,a'\,bf(A)],\end{aligned}$$

hence it suffices to prove that (9) coincides with (10). But

$$a\cup f(O)=f(O)\cup a\cdot b\cdot(a\cup b)=f(O)\cup\Pi_\Xi(\xi_1\cup\cdots\cup\xi_n),$$

$$b'\cup f(I)=f(I)\cup a'\cdot b'\cdot(a'\cup b')=f(I)\cup\Pi_\Xi(\xi_1'\cup\cdots\cup\xi_n'),$$

$$f(A)\cup a\cup b'=f(A)\cup 1\cdot(a\cup b')\cdot(a'\cup b)=$$

$$=f(A)\cup\Pi_\Xi(\xi_1^{\alpha_1'}\cup\cdots\cup\xi_n^{\alpha_n'})\ (\forall A\neq O, I),$$

$$a'f(O)=f(O)(a'\cup b'\cup a'b')=f(O)\cup_{\Xi}\xi_1'\ldots\xi_n',$$
$$bf(I)=f(I)(a\cup b\cup ab)=f(I)\cup_{\Xi}\xi_1\ldots\xi_n,$$
$$a'bf(A)=f(A)(0\cup ab'\cup a'b)=f(A)\cup_A\Xi^A\ (\forall\ A\neq O,\ I),$$

so that (10) can be written

$$[\Pi_A[f(A)\cup\Pi_{\Xi}(\xi_1^{\alpha_1'}\cup\cdots\cup\xi_n^{\alpha_n'})],$$
$$\cup_A f(A)\cup_{\Xi}\Xi^A]=[\Pi_A\Pi_{\Xi}[f(A)\cup\xi_1^{\alpha_1'}\cup\cdots\cup\xi_n^{\alpha_n'}],$$
$$\cup_A\cup_{\Xi}f(A)\,\Xi^A],$$

which coincides with (9).

Theorem 2. *The Boolean transformation* (1) *has the range* $\mathbf{G}\subseteq\mathbf{B}^m$ *defined by a Boolean equation* $g(Y)=1$ *if and only if*

$$\cup_C[g(C)+\cup_A d(A)F^C(A)]=0. \tag{11}$$

Proof. The equation (2) coincides with $g(Y)=1$ if and only if

$$\cup_A d(A)F^C(A)=g(C)\qquad(\forall\ C\in\{0,1\}^m), \tag{12}$$

which coincides with (11).

Corollary. *The Boolean transformation* $F:\mathbf{B}^n\to\mathbf{B}^m$ *has the range* $G\subseteq B^m$ *defined by a Boolean equation* $g(Y)=1$ *if and only if*

$$\cup_A F^C(A)=g(C)\qquad(\forall\ C\in\{0,1\}^m). \tag{13}$$

Proof. Take $d\equiv 1$ in (12).

Theorem 3. *The Boolean transformation* (1) *is a constant mapping if and only if*

$$\cup_C\{\Pi_A[d'(A)\cup F^C(A)]\}\{\cup_A d(A)F^C(A)\}=1. \tag{14}$$

Proof. Using the notations from theorem 2, $F$ is constant if and only if $\mathbf{G}$ is a singleton, that is, if and only if the equation $g(Y)=1$ has a unique solution. According to a theorem of Whitehead [8] (see also [5], theorem 6.7), the latter condition holds if and only if $\{g(C)\}_C$ is an orthonormal system. Therefore $F$ is constant if and only if equation (11) has an orthonormal solution with respect to the $2^m$ unknowns $g(C)$, indexed by $C\in\{0,1\}^m$.

Setting $h(\{g(C)\}_C)$ for the left-hand side of (11), the necessary and sufficient condition given by Johnson [3] (see also [5], theorem 4.7) for the existence of an orthonormal solution to (11) becomes

$$\Pi_D h(\Delta_D)=0, \tag{15}$$

where $\Delta_D\in\{0,1\}^m$ is the vector for which the coordinate of rank $D$ is 1, the other coordinates being 0. Now (15) can be written successively in the following equivalent forms:

$$\Pi_D\{\cup_{C\neq D}\cup_A d(A)F^C(A)\cup[\cup_A d(A)F^D(A)]'\}=0,$$
$$\Pi_D\{\cup_A d(A)[F^D(A)]'\cup[\cup_A d(A)F^D(A)]'\}=0,$$

the latter condition coincides with (14) because

$$\bigcup_{C \neq D} F^C(A) = [F^D(A)]'$$

in view of the orthonormality of the system $\{F^C(A)\}_C$ (cf. Löwenheim; see also [5], corollary 1 of proposition 4.3).

Corollary. *The Boolean transformation* $F: B^n \to \mathbf{B}^m$ *is a constant if and only if*

(16) $$\bigcup_C \Pi_A F^C(A) = 1.$$

Proof. Take $d \equiv 1$ in theorem 3.

Theorem 4. *The Boolean transformation* (1) *takes the constant value* $G \in \mathbf{B}^m$ *if and only if*

(17) $$\bigcup_C [G^C + \bigcup_A d(A) F^C(A)] = 0.$$

Proof. Using the notations from theorem 2, $F$ takes the constant value $G$ if and only if $\mathbf{G}$ reduces to the singleton $\{G\}$, that is, if and only if the equation $g(Y) = 1$ has the unique solution $Y := G$. According to a theorem of Bernstein [1] (see also [5]. theorem 6.6), the latter condition holds if and only if $g(C) = G^C$, so that equation (11) becomes (17).

Corollary. *The Boolean transformation* $F: \mathbf{B}^n \to \mathbf{B}^m$ *takes the constant value* $G \in \mathbf{B}^m$ *if and only if*

(18) $$\bigcup_A F^C(A) = G^C \qquad (\forall\, C \in \{0, 1\}^m).$$

Comment. A well known property of orthonormal systems shows readily that $F \equiv G$ if and only if $F^C(A) = G^C$ for every $A$ and $C$. Therefore the sufficiency of the seemingly weaker condition (18) is the non-trivial part of this corollary.

Proof. Take $d \equiv 1$ in theorem 4.

## 2. A few particular cases

*Case* $m := 1$

(2) $$[\bigcup_A d(A) f(A)]\, y \cup [\bigcup_A d(A) f'(A)]\, y' = 1,$$

(4) $$[\bigcup_A d(A) f(A)]\,[\bigcup_A d(A) f'(A)] = 1,$$

(11) $$[g(1) + \bigcup_A d(A) f(A)] \cup [g(0) + \bigcup_A d(A) f'(A)] = 0,$$

(13) $$\bigcup_A f(A) = g(1), \quad \bigcup_A f'(A) = g(0),$$

(14) $$\{\Pi_A [d'(A) \cup f(A)]\}\{\bigcup_A d(A) f(A)\} \cup$$
$$\cup \{\Pi_A [d'(A) \cup f'(A)]\}\{\bigcup_A d(A) f'(A)\} = 1,$$

(16) $$\Pi_A f(A) \cup \Pi_A f'(A) = 1,$$

(17) $$[g + \bigcup_A d(A) f(A)] \cup [g' + \bigcup_A d(A) f'(A)] = 0,$$

(18) $$\bigcup_A f(A) = g, \quad \bigcup_A f'(A) = g'.$$

*Case* $m:=2$

$$(2)\quad \begin{aligned}&[\cup_A d(A)f_1(A)f_2(A)]\,y_1y_2\cup[\cup_A d(A)f_1(A)f_2'(A)]\,y_1y_2'\cup\\ &\cup[\cup_A d(A)f_1'(A)f_2(A)]\,y_1'y_2\cup[\cup_A d(A)f_1'(A)f_2'(A)]\,y_1'y_2'=1,\end{aligned}$$

$$(4)\quad \begin{aligned}&[\cup_A d(A)f_1(A)f_2(A)]\,[\cup_A d(A)f_1(A)f_2'(A)]\cdot\\ &\cdot[\cup_A d(A)f_1'(A)f_2(A)]\,[\cup_A d(A)f_1'(A)f_2'(A)]=1,\end{aligned}$$

$$(11)\quad \begin{aligned}&[g(1,1)+\cup_A d(A)f_1(A)f_2(A)]\cup[g(1,0)+\cup_A d(A)f_1(A)f_2'(A)]\cup\\ &\cup[g(0,1)+\cup_A d(A)f_1'(A)f_2(A)]\cup[g(0,0)+\cup_A d(A)f_1'(A)f_2'(A)]=0,\end{aligned}$$

$$(13)\quad \begin{aligned}&\cup_A f_1(A)f_2(A)=g(1,1),\quad \cup_A f_1(A)f_2'(A)=g(1,0),\\ &\cup_A f_1'(A)f_2(A)=g(0,1),\quad \cup_A f_1'(A)f_2'(A)=g(0,0),\end{aligned}$$

$$(14)\quad \begin{aligned}&\{\Pi_A[d'(A)\cup f_1(A)f_2(A)]\}\{\cup_A d(A)f_1(A)f_2(A)\}\cup\\ &\cup\{\Pi_A[d'(A)\cup f_1(A)f_2'(A)]\}\{\cup_A d(A)f_1(A)f_2'(A)\}\cup\\ &\cup\{\Pi_A[d'(A)\cup f_1'(A)f_2(A)]\}\{\cup_A d(A)f_1'(A)f_2(A)\}\cup\\ &\cup\{\Pi_A[d'(A)\cup f_1'(A)f_2'(A)]\}\{\cup_A d(A)f_1'(A)f_2'(A)\}=1,\end{aligned}$$

$$(16)\quad \Pi_A f_1(A)f_2(A)\cup\Pi_A f_1(A)f_2'(A)\cup\Pi_A f_1'(A)f_2(A)\cup\Pi_A f_1'(A)f_2'(A)=1,$$

$$(17)\quad \begin{aligned}&[g_1g_2+\cup_A d(A)f_1(A)f_2(A)]\cup[g_1g_2'+\cup_A d(A)f_1(A)f_2'(A)]\cup\\ &\cup[g_1'g_2+\cup_A d(A)f_1'(A)f_2(A)]\cup[g_1'g_2'+\cup_A d(A)f_1'(A)f_2'(A)]=0,\end{aligned}$$

$$(18)\quad \begin{aligned}&\cup_A f_1(A)f_2(A)=g_1g_2,\quad \cup_A f_1(A)f_2'(A)=g_1g_2',\\ &\cup_A f_1'(A)f_2(A)=g_1'g_2,\quad \cup_A f_1'(A)f_2'(A)=g_1'g_2'.\end{aligned}$$

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