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DEGREE OF CONVERGENCE OF QUASI-HERMITE-FEJÉR INTERPOLATION

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Let

$$-1=x_{n+1}<x_n<\cdots<x_1<x_0=1$$

be $n+2$ given points and let $f(x)$ be a given function on $[-1, 1]$. P. Szász [3] introduced the interpolation polynomials $R_n(f; x)$ of degree $\leqslant 2n+1$ satisfying the conditions:

$$R_n(f; x_\nu)=f(x_\nu); \quad \nu=0,\ 1,\ 2, \ldots, n+1,$$

$$R_n'(f; x_\nu)=0; \quad \nu=1,\ 2, \ldots, n.$$

He called $R_n(f; x)$ – the quasi-Hermite-Fejér interpolation polynomials. For the explicit expression of $R_n(f; x)$ we have

$$\begin{aligned}(1.1)\qquad R_n(f; x)=&f(-1)\frac{(1-x)\,w(x)^2}{2w(-1)^2}+f(1)\frac{(1+x)\,w(x)^2}{2w(1)^2}\\ &+\sum_{\nu=1}^{n} f(x_\nu)\frac{1-x^2}{1-x_\nu^2}[1+c_\nu(x-x_\nu)]\left(\frac{w(x)}{(x-x_\nu)\,w'(x_\nu)}\right)^2\end{aligned}$$

where

$$w(x)=C\prod_{\nu=1}^{n}(x-x_\nu), \quad C\neq 0$$

and

$$c_\nu=\frac{2x_\nu}{1-x_\nu^2}-\frac{w''(x_\nu)}{w'(x_\nu)}; \quad \nu=1,\ 2, \ldots, n.$$

If $w(x)=P_n(x)$ where $P_n(x)$ is the Legendre polynomial of exact degree n then from (1.1) we have

$$(1.2)\qquad R_n(f; x)=f(-1)\frac{1-x}{2}P_n^2(x)+f(1)\frac{1+x}{2}P_n^2(x)+\sum_{\nu=1}^{n} f(x_\nu)A_\nu(x)$$

where

$$(1.3)\qquad A_\nu(x)=\frac{1-x^2}{1-x_\nu^2}\left(\frac{P_n(x)}{(x-x_\nu)\,P_n'(x_\nu)}\right)^2.$$

The polynomials (1.2) were first obtained by Egerváry and P. Turán [1] as the solution of the problem of most economical interpolation process. They showed that for $f(x) \in C[-1, 1]$

$$\lim_{n \to \infty} R_n(f; x) = f(x). \tag{1.4}$$

A little differently from Egerváry and Turán, P. Szász also established the same result using the non-negativity of the fundamental polynomials. Utilizing the same fact we can find the degree of convergence of the sequence $R_n(f; x)$ in a simple way, namely the theorem 1 below. Let us denote by $C_\omega[-1, 1]$ the class of all those functions defined on $[-1, 1]$ for which

$$\omega_f(\delta) \leqslant c\,\omega(\delta) \tag{1.5}$$

where $\omega_f(\delta)$ is the modulus of continuity of $f(x)$ and $\omega(\delta)$ is a certain modulus of continuity. We shall prove the following:

Theorem 1. *If $f(x) \in C_\omega[-1, 1]$ then*

$$|R_n(f; x) - f(x)| \leqslant K_1 \omega\left(\frac{(1-x^2)^{1/4}}{\sqrt{n}}\right) \tag{1.6}$$

where K_1 (later on $K_2, K_3, \ldots$) is an absolute constant.

The result (1.6) appears to be interesting in the sense that $f(x)$ can be approximated arbitrarily at the end points of $[-1, 1]$ by $R_n(f; x)$. As for the order is concerned a better result than (1.6) on the pattern of Vértesi [5] can be established. Precisely the following is true:

Theorem 2. *If $f(x) \in C_\omega[-1, 1]$ then for all $x \in [-1, 1]$*

$$|R_n(f; x) - f(x)| \leqslant K_2 \left[\sum_{i=1}^{n} \frac{1}{i^2}\, \omega\left(\frac{i\sqrt{1-x^2}}{n}\right) + \sum_{i=1}^{n} \frac{1}{i^2}\, \omega\left(\frac{i^2}{n^2}\right)\right]. \tag{1.7}$$

From (1.5) and (1.7) it follows that

$$|R_n(f; x) - f(x)| \leqslant K_3 \left[\sum_{i=1}^{n} \frac{1}{i^2}\, \omega_f\left(\frac{i\sqrt{1-x^2}}{n}\right) + \sum_{i=1}^{n} \frac{1}{i^2}\, \omega_f\left(\frac{i^2}{n^2}\right)\right]. \tag{1.8}$$

Now if $\omega_f(\delta) \leqslant \delta^\alpha \,(0 < \alpha < 1)$ then from (1.8) we have that

$$|R_n(f; x) - f(x)| \leqslant K_4 \left[\frac{(1-x^2)^{\alpha/2}}{n^\alpha} + \frac{1}{n^{2\alpha}}\right], \quad (0 < \alpha < 1/2; -1 \leqslant x \leqslant 1), \tag{1.9}$$

$$|R_n(f; x) - f(x)| \leqslant K_5 \left[\frac{(1-x^2)^{\alpha/2}}{n^\alpha} + \frac{\log n}{n}\right], \quad (\alpha = 1/2; -1 \leqslant x \leqslant 1) \tag{1.10}$$

and

$$|R_n(f; x) - f(x)| \leqslant K_6 \left[\frac{(1-x^2)^{\alpha/2}}{n^\alpha} + \frac{1}{n}\right], \quad (1/2 < \alpha < 1; -1 \leqslant x \leqslant 1). \tag{1.11}$$

Moreover, if $\omega(\delta) = \delta$ then

$$|R_n(f; x) - f(x)| \leqslant K_7 \left[\sqrt{1-x^2}\left(\frac{\log n}{n}\right) + \frac{1}{n}\right], \quad (-1 \leqslant x \leqslant 1). \tag{1.12}$$

Our result is best possible for $f(x)\in \mathrm{Lip}\,\alpha\,(0<\alpha<1/2)$ when $-1<x<1$. For the case when $f(x)\in \mathrm{Lip}\,1$ and $-1<x<1$ we shall prove the following:

Theorem 3. *There exists a function* $f(x)\in \mathrm{Lip}\,1$ *and a constant* c^* *such that*

$$(1.13)\qquad |R_n(f;\,0)-f(0)|\geqslant c^*\frac{\log n}{n},\quad n=6,\,8,\,10,\,\dots .$$

Further making use of the inequalities [6]

$$(1.14)\qquad \frac{\pi}{n+1}\int\limits_{\frac{\pi}{n+1}}^{\pi}\frac{\omega\left(t\sqrt{1-x^2}\right)}{t^2}\,dt\leqslant\sum_{i=1}^{n}\frac{1}{i^2}\,\omega\left(\frac{i+1}{n+1}\pi\sqrt{1-x^2}\right)\leqslant$$

$$\leqslant\frac{8\pi}{n+1}\int\limits_{\frac{\pi}{n+1}}^{\pi}\frac{\omega\left(t\sqrt{1-x^2}\right)}{t^2}\,dt,$$

$$(1.15)\qquad \frac{\pi}{n+1}\int\limits_{\frac{\pi}{n+1}}^{\pi}\frac{\omega(t^2)}{t^2}\,dt\leqslant\sum_{i=1}^{n}\frac{1}{i^2}\,\omega\left(\frac{(i+1)^2\pi^2}{(n+1)^2}\right)\leqslant\frac{8\pi}{n+1}\int\limits_{\frac{\pi}{n+1}}^{\pi}\frac{\omega(t^2)}{t^2}\,dt$$

and following on the lines of [6] the estimates in Theorem 2 can be obtained in terms of the arithmetic means of $\left\{\omega\left(\frac{\sqrt{1-x^2}}{n}+\frac{1}{n^2}\right)\right\}$ as given by the following:

Theorem 4. *If* $f(x)\in C_\omega[-1,\,1]$ *then for all* $x\in[-1,\,1]$,

$$(1.16)\qquad |R_n(f;\,x)-f(x)|\leqslant K_8\,n^{-1}\sum_{\nu=1}^{n}\omega\left(\frac{\sqrt{1-x^2}}{\nu}+\frac{1}{\nu^2}\right).$$

The inequality (1.16) is an improvement over that of Mills and Varma [2] in the sense that it gives better local approximation at the end points of the interval.

2. We recall the following well-known results which will be of frequent use. From [1] we have for $-1\leqslant x\leqslant 1$ and $n=1,\,2,\,3,\,\dots$

$$(2.1)\qquad \sum_{\nu=1}^{n}\frac{1-x^2}{1-x_\nu^2}\left(\frac{P_n(x)}{(x-x_\nu)\,P_n'(x_\nu)}\right)^2\equiv 1-P_n^2(x)\leqslant 1.$$

From Szegö [4] we have for $-1\leqslant x\leqslant 1$,

$$(2.2)\qquad |P_n(x)|\leqslant 1,$$

$$(2.3)\qquad (1-x^2)^{1/4}\,|P_n(x)|\leqslant(2/\pi)^{1/2}\,n^{-1/2},$$

$$(2.4)\qquad |P_n'(x_\nu)|\sim\nu^{-3/2}n^2,\quad \nu=1,\,2,\,\dots,\,\left[\frac{n}{2}\right],$$

(2.5) $$|P_n'(x_\nu)| \sim (n+1-\nu)^{-3/2} n^2, \quad \nu = \left[\frac{n}{2}\right]+1, \ldots, n,$$

(2.6) $$(1-x_\nu^2) > (\nu - 1/2)^2 (n+1/2)^{-2}, \quad \nu = 1, 2, \ldots, \left[\frac{n}{2}\right],$$

(2.7) $$(1-x_\nu^2) > (n-\nu+1/2)^2 (n+1/2)^{-2}, \quad \nu = \left[\frac{n}{2}\right]+1, \ldots, n,$$

and

(2.8) $$\frac{(\nu-1/2)\pi}{n+1/2} < \theta_\nu < \frac{\nu\pi}{n+1/2} \quad \nu = 1, 2, \ldots, n,$$

$x_\nu = \cos\theta_\nu$ and $x = \cos\theta$.

Also from Szász [3] we have that

(2.9) $$\sum_{\nu=1}^{n} \frac{1}{(1-x_\nu^2)[P_n'(x_\nu)]^2} = 1.$$

3. We denote by x_j the zero of $P_n(x)$ which is nearest to x; then as in [5] we can prove that

(3.1) $$|f(x)-f(x_\nu)| \leqslant K_9\left[\omega\left(\frac{\sin\theta}{n}\right)+\omega\left(\frac{1}{n^2}\right)\right] \text{ if } \nu = j$$

$$\leqslant K_{10}\left[\omega\left(\frac{i\sin\theta}{n}\right)+\omega\left(\frac{i^2}{n^2}\right)\right] \text{ if } j<\nu=j+i\leqslant n$$

$$\text{or } 1\leqslant \nu = j-i<j.$$

Proof of Theorem 1. From (1.2) and (2.1) we have that

(3.2) $$R_n(f;x)-f(x) = \left\{[f(1)-f(x)]\frac{1+x}{2}+[f(-1)-f(x)]\frac{1-x}{2}\right\}P_n^2(x) + \sum_{\nu=1}^{n}[f(x_\nu)-f(x)]A_\nu(x)$$

and hence

(3.3) $$|R_n(f;x)-f(x)| \leqslant \frac{1}{2}[(1+x)\,\omega(1-x)+(1-x)\,\omega(1+x)]P_n^2(x) + \sum_{\nu=1}^{n}\omega(|x-x_\nu|)A_\nu(x) = S_1+S_2.$$

Now consider

$$S_1 = \frac{1}{2}[(1+x)\,\omega(1-x)+(1-x)\,\omega(1+x)]P_n^2(x).$$

Let $x \geqslant 0$ then

$$\frac{\omega(1+x)}{1+x} \leqslant \frac{2\,\omega(1-x)}{1-x}$$

and hence

$$S_1 \leqslant \frac{3}{2}(1+x)\,\omega(1-x)\,P_n^2(x) \leqslant 3\,\omega(1-x^2)\,P_n^2(x)$$

which is also valid if $x<0$. So it follows that for $-1 \leqslant x \leqslant 1$,

$$S_1 \leqslant 3\,\omega(1-x^2)\,P_n^2(x).$$

Now making use of (2.2), (2.3) and simplifying we obtain

$$S_1 \leqslant 3\,[n\sqrt{1-x^2}+1]\,\omega\left(\frac{\sqrt{1-x^2}}{n}\right) P_n^2(x) \leqslant 6\,\omega\left(\frac{\sqrt{1-x^2}}{n}\right). \tag{3.4}$$

Owing to (2.1) we have that

$$S_2 = \sum_{\nu=1}^{n} \omega(|x-x_\nu|)\,A_\nu(x) \leqslant \omega(\delta) \sum_{\nu=1}^{n} \left[1+\frac{|x-x_\nu|}{\delta}\right] A_\nu(x)$$

$$= \omega(\delta) \sum_{\nu=1}^{n} A_\nu(x) + \frac{\omega(\delta)}{\delta} \sum_{\nu=1}^{n} |x-x_\nu|\,A_\nu(x)$$

$$\leqslant \omega(\delta) + \frac{\omega(\delta)}{\delta} \sum_{\nu=1}^{n} |x-x_\nu|\,A_\nu(x). \tag{3.5}$$

On applying Schwarz's inequality and using (2.1), (2.9) and (2.3) we get

$$\sum_{\nu=1}^{n} |x-x_\nu|\,A_\nu(x) \leqslant \left[\sum_{\nu=1}^{n} (x-x_\nu)^2 A_\nu(x)\right]^{1/2} \left[\sum_{\nu=1}^{n} A_\nu(x)\right]^{1/2}$$

$$\leqslant \left[\sum_{\nu=1}^{n} (x-x_\nu)^2 A_\nu(x)\right]^{1/2}$$

$$= \sqrt{1-x^2}\,|P_n(x)| \left[\sum_{\nu=1}^{n} \frac{1}{(1-x_\nu^2)\{P_n'(x_\nu)\}^2}\right]^{1/2} = \sqrt{1-x^2}\,|P_n(x)|$$

$$\leqslant (2/\pi)^{1/2}(1-x^2)^{1/4}\,n^{-1/2}. \tag{3.6}$$

Consequently from (3.5) and (3.6) we have that

$$S_2 \leqslant 2\,\omega\left(\frac{(1-x^2)^{1/4}}{\sqrt{n}}\right) \tag{3.7}$$

if we choose $\delta = n^{-1/2}(1-x^2)^{1/4}$.
Hence from (3.3). (3.4) and (3.7) the theorem follows.

Proof of Theorem 2. (3.2) yields

$$|R_n(f;x)-f(x)| \leqslant \left[\frac{1+x}{2}|f(-1)-f(x)| + \frac{1-x}{2}|f(-1)-f(x)|\right] P_n^2(x) \tag{3.8}$$

$$+ \sum_{\nu=1}^{n} |f(x_\nu)-f(x)|\,A_\nu(x) = \sigma_1 + \sigma_2.$$

As above in this case also we get

$$\sigma_1 \leqslant 6\,\omega\left(\frac{\sqrt{1-x^2}}{n}\right). \tag{3.9}$$

Further we have

$$\sigma_2 = \sum_{\nu=1}^{n} |f(x_\nu)-f(x)|\,A_\nu(x) \tag{3.10}$$

$$= |f(x_j)-f(x)|\,A_j(x) + \sum_{\nu\neq j} |f(x_\nu)-f(x)|\,A_\nu(x) = \sigma_1^* + \sigma_2^*.$$

Owing to (2.1) and (3.1) we obtain

$$\sigma_1^* \leqslant K_9\left[\omega\left(\frac{\sin\theta}{n}\right)+\omega\left(\frac{1}{n^2}\right)\right]A_j(x) \tag{3.11}$$

$$\leqslant K_9\left[\omega\left(\frac{\sin\theta}{n}\right)+\omega\left(\frac{1}{n^2}\right)\right].$$

If $x=\cos\theta$ and $x_\nu=\cos\theta_\nu$ then (2.6) and (2.7) yield

$$\frac{\sqrt{1-x^2}}{(x-x_\nu)^2}=\frac{\sin\theta}{(\cos\theta-\cos\theta_\nu)^2}\leqslant\frac{1}{2\sin^2\left(\frac{\theta-\theta_\nu}{2}\right)\sin\left(\frac{\theta+\theta_\nu}{2}\right)} \tag{3.12}$$

$$\leqslant\frac{1}{\sin^2\left(\frac{\theta-\theta_\nu}{2}\right)\sin\theta_\nu}\leqslant\frac{(2n+1)}{(2\nu-1)\sin^2\left(\frac{\theta-\theta_\nu}{2}\right)}.$$

From (2.8) and using the fact that x_j is the nearest zero to x we get

$$\frac{1}{\sin\frac{|\theta-\theta_\nu|}{2}}\leqslant\frac{2n+1}{2i-1},\quad \nu\neq j;\ \nu=j\pm i. \tag{3.13}$$

Consequently from (3.12) (3.13), (2.3), (2.4), (2.5), (2.6) and (2.7) it follows that

$$A_\nu(x)\leqslant\frac{K_{11}}{i^2},\quad \nu\neq j,\ \nu=j\pm i. \tag{3.14}$$

Now making use of (3.14) and (3.1) we obtain

$$\sigma_2^*\leqslant K_{12}\sum_{\nu\neq j}\frac{1}{i^2}\left[\omega\left(\frac{i\sqrt{1-x^2}}{n}\right)+\omega\left(\frac{i^2}{n^2}\right)\right]. \tag{3.15}$$

So from (3.10), (3.11) and (3.15) it follows that for $-1\leqslant x\leqslant 1$,

$$\sigma_2\leqslant K_9\left[\omega\left(\frac{\sqrt{1-x^2}}{n}\right)+\omega\left(\frac{1}{n^2}\right)+K_{12}\sum_{\nu\neq j}\frac{1}{i^2}\left[\omega\left(\frac{i\sqrt{1-x^2}}{n}\right)+\omega\left(\frac{i^2}{n^2}\right)\right]\right. \tag{3.16}$$

Thus from (3.8), (3.9) and (3.16) we have the theorem.

4. Proof of Theorem 3. Let $f(x)=|x|$, $x=\cos\theta=0$, $\theta=\frac{\pi}{2}$ and $n=6, 8, 10, \ldots$. Then from (1.2) and (2.1) it follows that

$$R_n(f;0)-f(0)=P_n^2(0)+\sum_{\nu=1}^{n}[f(x_\nu)-f(0)]A_\nu(0) \tag{4.1}$$

$$=P_n^2(0)+\sum_{\nu=1}^{n}\frac{P_n^2(0)}{(1-x_\nu^2)|x_\nu|[P_n'(x_\nu)]^2}$$

$$\geqslant\sum_{\nu=1}^{\frac{n}{2}}\frac{P_n^2(0)}{x_\nu(1-x_\nu^2)[P_n'(x_\nu)]^2}.$$

From Szegö [4], p. 163 we have that

$$|P_n(0)|=\frac{1\cdot3\cdot5\cdot\cdots\cdot(n-1)}{2\cdot4\cdot\cdots\cdot(n-2)n} \tag{4.2}$$

$$>\frac{1}{3}n^{-1/2}.$$

Now on using (4.2), (2.8) and (2.4) we get from (4.1),

$$R_n(f;0)-f(0)\geqslant K_{13}n^{-3}\sum_{\nu=1}^{\frac{n}{2}}\frac{\nu}{\cos\theta_\nu}. \tag{4.3}$$

Further, if $\theta=\pi/2$ then on using (2.8) we obtain

$$\sum_{\nu=1}^{\frac{n}{2}}\frac{\nu}{\cos\theta_\nu}=\sum_{\nu=1}^{\frac{n}{2}}\frac{\nu}{(\cos\theta_\nu-\cos\theta)} \tag{4.4}$$

$$=\sum_{\nu=1}^{\frac{n}{2}}\frac{\nu}{2\sin\left(\frac{\theta+\theta_\nu}{2}\right)\sin\left(\frac{\theta-\theta_\nu}{2}\right)}\geqslant\sum_{\nu=1}^{\frac{n}{2}}\frac{\nu}{2\left(\frac{\theta-\theta_\nu}{2}\right)}$$

$$\geqslant\frac{2n}{\pi}\sum_{\nu=1}^{\frac{n}{2}}\frac{\nu}{n-2\nu+2}>K_{14}n^2\log n,\quad n\geqslant6.$$

Hence from (4.3) and (4.4) it follows that

$$R_n(f;0)-f(0)>\frac{c^*\log n}{n} \tag{4.5}$$

from which the Theorem follows.

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