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ON THE SPHERICAL MEANS AND SOME OF THEIR APPLICATIONS*

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1. Let $X(x_1,\ldots,x_n)$ be a point in the space E^n and $f(X)=f(x_1,\ldots,x_n)$ a real valued, L-integrable function having the period 2π in each variable. Let

$$f(X)\sim\sum_{-\infty}^{+\infty}\cdots\sum_{-\infty}^{+\infty} a_{m_1\ldots m_n}\exp\left(i\sum_{j=1}^{n} m_j x_j\right) \tag{1}$$

be its multiple Fourier series. The functions $\exp\left(i\sum_{j=1}^{n} m_j x_j\right)$, where m_j ($j=1, 2,\ldots,n$) are integers, represent a complete set of regular solutions of the characteristic value problem

$$\Delta u(X)+\lambda u(X)=0$$

in the domain $0\leqslant x_j<2\pi$, $j=1, 2,\ldots,n$ with the characteristic values $\lambda=m_1^2+\cdots+m_n^2$, where Δ is the Laplace differential operator.

In this article we shall prove two theorems: Theorem 1. and Theorem 2.

We shall use the following notations, definitions and results:

(i) The spherical partial sums of order k of series (1) is defined by

$$\sigma_k(X)=\sum_{\nu=0}^{k} A_\nu(X), \tag{2}$$

where

$$A_\nu(X)=\sum_{m_1^2+\cdots+m_n^2=\nu} a_{m_1\ldots m_n}\exp\left(i\sum_{j=1}^{n} m_j x_j\right),$$

and $A_\nu(X)\equiv 0$ if k cannot be represented as a sum of n squares.

(ii) Φ-mean of spherical partial sums (2) is defined [2] by

$$S_w^{\Phi}(X)=\sum_{\mu\leqslant w}\Phi\left(\frac{\mu}{w}\right)a_{m_1\ldots m_n}\exp\left(i\sum_{j=1}^{n} m_j x_j\right), \tag{3}$$

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where $\mu^2 = m_1^2 + \cdots + m_n^2$, and $\Phi(t)$ is a function defined for $a \leqslant t < \infty$, $\Phi(0) = 1$. $S_w^{\Phi}(X)$ can be expressed [2] by

$$S_w^{\Phi}(X) = 2^{1-n/2} [\Gamma(n/2)]^{-1} w \int_0^{\infty} f(X; t) H_{\Phi}(wt)\, dt, \tag{4}$$

whenever

$$\int_0^{\infty} |\Phi(t)|\, t^{n-1}\, dt < \infty. \tag{5}$$

(iii) $f(X; t)$ is the spherical mean of the function $f(X)$ over a sphere whose radius is t and its centre is at the point $X(x_1, \ldots, x_n)$, i.e.

$$f(X; t) = 2^{-1} \pi^{-n/2} \Gamma(n/2) \int_{\sigma} f(x_1 + t\,\xi_1, \ldots, x_n + t\,\xi_n)\, d\sigma_{\xi}, \tag{6}$$

where σ is the unit sphere $\xi_1^2 + \cdots + \xi_n^2 = 1$, and $d\sigma_{\xi}$ its $(n-1)$-dimensional volume element.

(iv) The kernel $H_{\Phi}(v)$ is defined [2] by

$$H_{\Phi}(v) = v^{-1} \int_0^{\infty} \Phi\left(\frac{u}{v}\right) u^{n-1} V_{n/2-1}(u)\, du = v^{n-1} \int_0^{\infty} \Phi(u)\, u^{n-1} V_{n/2-1}(vu)\, du, \tag{7}$$

where

$$V_{\mu}(u) = u^{-\mu} J_{\mu}(u),$$

and $J_{\mu}(u)$ is the Bessel function of the first kind of order μ.

(v) We assume that $\Phi(t)$ has the following properties [2]:

(a) The inequality (5) holds.

(b) If r is the integer defined by $-1/2 \leqslant n/2 - 1 - r < 1/2$, then $\Phi(t)$ has $(r+2)$ derivatives in $0 \leqslant t < \infty$, each bounded in some neighborhood of $t = 0$, such that

$$\limsup |\Phi^{(\mu)}(u)\, u^{\gamma}| < \infty, \quad \int_0^{\infty} |\Phi^{(\mu)}(u)\, u^{\gamma}|\, du < \infty$$

for $\mu = 0, 1, 2, \ldots, r+2$; $0 \leqslant \gamma < (n-1)/2$.

Lemma A. *If λ is any real number $\geqslant -1/2$ and if $\Phi(t)$ satisfies property (b), then* [2]

$$\int_0^{\infty} \Phi(u)\, u^{2\lambda+1} V_{\lambda}(vu)\, du = 0\,(v^{-2\lambda-2-\alpha}) \text{ for } \alpha > 0, \text{ as } v \to \infty. \tag{8}$$

(vi) The spherical means of order s of the function $f(X)$ are defined [3] by

$$f_s(X; t) = 2^s \Gamma(s) [B(s, n/2)]^{-1} t^{-n+2-2s} \psi_s(X; t), \text{ for } s > 0 \tag{9}$$
$$= f(X; t), \text{ for } s = 0,$$

where

$$\psi_s(X; t) = 2^{1-s} [\Gamma(s)]^{-1} \int_0^t (t^2 - \tau^2)^{s-1} \tau^{n-1} f(X; \tau)\, d\tau, \; s > 0.$$

(vii) A particular case of Φ-mean (3) is the Riesz mean $S_w^k(X)$ of order $k\,(k>0)$ of spherical partial sums (2) defined [3] by

$$(10)\quad S_w^k(X)=\sum_{m_1^2+\cdots+m_n^2\leq w^2}\{1-(m_1^2+\cdots+m_n^2)^k\,w^{-2}\}\,a_{m_1\ldots m_n}\exp\Big(i\sum_{j=1}^n m_j\,x_j\Big).$$

There is the following relation [3] between $f_s(X;t)$ and $S_w^k(X)$:

$$(11)\qquad f_s(X;t)=c_1\,t^{2k+2}\int_0^\infty S_w^k(X)\,w^{2k+1}\,V_{s+k+n/2}(tw)\,dw,$$

where

$$s>1,\ k>(n-1)/2 \text{ and } c_1=2^{s-1-k+n/2}\,\Gamma(s+n/2)\,[\Gamma(1+k)]^{-1}.$$

(viii) $L(x)$ belongs to the class of slowly oscillating functions at infinity if it is positive and continuous in $0\leqslant x<\infty$ and

$$\lim_{x\to\infty} L(tx)/L(x)=1 \text{ for every fixed } t>0.$$

We shall explore the following properties of slowly oscillating functions:

(α) If $\lambda>0$, then [4]

$$(12)\qquad x^{\lambda}L(x)\to\infty,\ x^{-\lambda}L(x)\to 0 \text{ as } x\to\infty.$$

(β) If $g(t)$ is such that both integrals

$$(13)\qquad \int_0^1 t^{-a}\,|g(t)|\,dt \text{ and } \int_1^\infty t^{a}|g(t)|\,dt$$

exist for some $a>0$, then [1]

$$(14)\qquad \int_0^\infty g(t)\,L(xt)\,dt\cong L(x)\int_0^\infty g(t)\,dt \quad\text{as } x\to\infty.$$

(ix) Throughout this article all c's, C's and M's denote the positive constants.

2. Now we shall consider Φ-summability of the derived multiple Fourier series obtained by successively applying s-times the Laplace differential operator to series (1) under the hypothesis that $t^{-2s}f(X;t)$ is of bounded variation.

By $\Delta_X^s[S_w^\Phi(X)]$ we denote that the Laplace operator is successively applied s-times to the Φ-mean of the spherical partial sums of series (1).

In the next theorem we shall meet the integral

$$(15)\qquad a=\int_0^\infty t^{n-1+2s}K_\Phi(t)\,dt,$$

where

$$(16)\qquad K_\Phi(t)=\sum_{\nu=0}^{s}(-1)^\nu c_\nu\,t^{2s-2\nu}H_\nu(t),$$

and

$$H_\nu(t)=\int_0^\infty \Phi(u)\, u^{4s-2\nu+n-1}\, V_{2s-\nu-1+n/2}(tu)\, du.$$

Now we will show that integral (15) does exist. In virtue of (9), Lemma A, we get

$$H_\nu(t)=0\,(t^{-n-4s+2\nu-\alpha}) \text{ as } t\to\infty,\ \alpha>0,$$

and according to (16) it comes out

(17) $$K_\Phi(t)=0\,(t^{-n-2s-\alpha}) \text{ as } t\to\infty,\ \alpha>0.$$

If we suppose that

(18) $$\int_0^\infty |\Phi(u)|\, u^{n-1+2s}\, du<\infty,$$

then we have

$$|H_s(t)|\leqslant\int_0^\infty |\Phi(u)|\, u^{n-1+2s}\, |V_{s-1+n/2}(tu)|\, du\leqslant M\int_0^\infty |\Phi(u)|\, u^{n-1+2s}\, du\leqslant M_1,$$

because

(19) $$|V_\mu(x)|\leqslant M \text{ on the interval } (0,\infty).$$

Exploring the results

(20) $$|V_\mu(x)|\leqslant M_1, \text{ for } 0\leqslant x\leqslant b$$

and

(21) $$|V_\mu(x)|\leqslant M_2\, x^{-\mu-1/2}, \text{ for } x\geqslant b,$$

where $\mu>-1$ and $b>0$, we shall estimate the integrals

$$t^{2s-2}H_\nu(t)=t^{2s-2\nu}\int_0^\infty \Phi(u)\, u^{4s-2\nu+n-1}\, V_{2s-\nu-1+n/2}(tu)\, du=$$

(22) $$=t^{2s-2}\left(\int_0^\omega+\int_\omega^{1/t}+\int_{1/t}^\infty\right)=H_{\nu1}(t)+H_{\nu2}(t)+H_{\nu3}(t),$$

$\nu=0,1,\dots,s-1$ and $\omega<1/t$.

According to (19) and (20) we obtain

$$|H_{\nu1}(t)|\leqslant t^{2s-2}\int_0^\omega |\Phi(u)|\, u^{4s-2\nu+n-1}\, V_{2s-\nu-1+n/2}(tu)\, du\leqslant$$

(23) $$\leqslant M_1\, t^{2s-2\nu}\int_0^\omega |\Phi(u)|\, u^{4s-2\nu+n-1}\, du\leqslant M_{11}\, t^{2s-2\nu},$$

and

$$| H_{\nu 2}(t) | \leqslant M_2 t^{2s-2\nu} \int_{\omega}^{1/t} | \Phi(u) | u^{4s-2\nu+n-1} du = M_2 \int_{\omega}^{1/t} | \Phi(u) | u^{2s+n-1} (tu)^{2s-2\nu} du \leqslant$$

$$\leqslant M_2 \int_{\omega}^{1/t} | \Phi(u) | u^{2s+n-1} du \leqslant M_{21}.$$

Further, with respect to (21) and (18) we get

$$| H_{\nu 3}(t) | \leqslant M_3 t^{2s-2\nu} \int_{1/t}^{\infty} | \Phi(u) | u^{4s-2\nu+n-1} (tu)^{-2s+\nu-(n-1)/2} du =$$

$$= M_3 \int_{1/t}^{\infty} | \Phi(u) | u^{2s+n-1} (tu)^{-\nu-(n-1)/2} du \leqslant M_{31}. \tag{25}$$

From (16), (22), (23), (24) and (25) it follows that for all finite values of t

$$| K_{\Phi}(t) | \leqslant C. \tag{26}$$

Finally, we can conclude that the integral

$$a = \int_0^{\infty} t^{n-1+2s} K_{\Phi}(t) dt = \int_0^{\delta} + \int_{\delta}^{\infty} = a_1 + a_2$$

exists, since a_1 exists because of (26), and according to (17)

$$a_2 = 0 \left(\int_{\delta}^{\infty} t^{-1-\alpha} dt \right) = 0(1).$$

Now we shall prove the following

Theorem 1. *Let $\Phi(t)$ have the property (b) and*

$$\int_0^{\infty} | \Phi(t) | t^{n-1+2s} dt < \infty, \tag{27}$$

where s is a nonnegative integer. If $\psi(X; t) = t^{-2s} f(X; t)$ is of bounded variation in $0 < t < \infty$, then

$$\lim_{w \to \infty} \Delta_X^s [S_w^{\Phi}(X)] = ca\, \psi(X; +o), \tag{28}$$

where $c = 2^{1-n/2} [\Gamma(n/2)]^{-1}$.

Proof. Substituting (6) in Bochner's fundamental formula (4) we get

$$S_w^{\Phi}(X) = (2\pi)^{-n/2} w \int_0^{\infty} \left\{ \int_{\sigma} f(X + t\xi) d\sigma_{\xi} \right\} H_{\Phi}(wt), dt, \tag{29}$$

where

$$X + t\xi = (x_1 + t\xi_1, \ldots, x_n + t\xi_n).$$

We shall introduce the polar coordinates $(R, \theta_1, \ldots, \theta_{n-1})$, $0 \leqslant R < \infty$, $0 \leqslant \theta_m \leqslant \pi$, $m = 1, 2, \ldots, n-2$ and $0 \leqslant \theta_{n-1} < 2\pi$. Since σ is the unit sphere, i.e. $R = 1$, we have

$$S_w^\Phi(X) = (2\pi)^{-n/2} w \int_0^\infty \int_0^\pi \cdots \int_0^\pi \int_0^{2\pi} f(x_1 + t\cos\theta_1, \ldots, x_n + t\sin\theta_1 \cdots \sin\theta_{n-1}) \cdot$$

$$\cdot \sin^{n-2}\theta_1 \sin^{n-3}\theta_2 \cdots \sin\theta_{n-2} H_\Phi(wt)\, dt\, d\theta_1 \cdots d\theta_{n-2}\, d\theta_{n-1}.$$

Now introducing the new coordinates η_j, $-\infty < \eta_j < \infty$, $j = 1, \ldots, n$ we obtain

$$S_w^\Phi(X) = (2\pi)^{-n/2} w \int_{-\infty}^\infty \cdots \int_{-\infty}^\infty f(X + \eta) |\eta|^{1-n} H_\Phi(w|\eta|)\, d\eta_1 \cdots d\eta_n =$$

$$= (2\pi)^{-n/2} w \int_{-\infty}^\infty \cdots \int_{-\infty}^\infty f(\eta) |X - \eta|^{1-n} H_\Phi(w|X - \eta|)\, d\eta_1 \cdots d\eta_n, \tag{30}$$

where

$$|\eta| = \sqrt{\eta_1^2 + \cdots + \eta_n^2}.$$

According to the formulae of transformation it follows that $t = |\eta|$.

Exploring the fact that we can differentiate Bochner's fundamental formula (4) we shall apply the Laplace operator Δ_X^s to the integral in (30) and then we get

$$I = (2\pi)^{-n/2} w \int_{-\infty}^\infty \cdots \int_{-\infty}^\infty f(\eta)\, \Delta_X^s \{|X - \eta|^{1-n} H_\Phi(w|X - \eta|)\}\, d\eta_1 \cdots d\eta_n =$$

$$= (2\pi)^{-n/2} w \int_{-\infty}^\infty \cdots \int_{-\infty}^\infty f(X + \xi)\, \Delta_\xi^s \{|\xi|^{1-n} H_\Phi(w|\xi|)\}\, d\xi_1 \cdots d\xi_n.$$

According to (7) we denote

$$F = F(t) = t^{1-n} H_\Phi(wt) = w^{n-1} \int_0^\infty \Phi(u)\, u^{n-1} V_{-1+n/2}(wtu)\, du,$$

where now

$$t = |\xi| = \sqrt{\xi_1^2 + \cdots + \xi_n^2}.$$

Using (6) and (29) we get

$$I = cw \int_0^\infty t^{n-1} (\Delta_t^s F) f(X; t)\, dt, \quad c = 2^{1-n/2} [\Gamma(n/2)]^{-1},$$

with

$$\Delta_t^s F = w^{n-1+2s} K_\Phi(wt),$$

where $K_\Phi(\tau)$ is defined by (16). The expression for $\Delta_t^s F$ is obtained [3] exploring the formula

$$\frac{d}{dx}[V_m(x)] = -x\, V_{m+1}(x).$$

Now we can write

$$I = cw^{n+2s} \int_0^\infty t^{n-1+2s} K_\Phi(wt)\, \psi(X;t)\, dt,$$

where

$$\psi(X;t) = t^{-2s} f(X;t),$$

or

$$I = cw^{n+2s} \psi(X;+0) \int_0^\infty t^{n-1+2s} K_\Phi(wt)\, dt + c\, w^{n+2s} \int_0^\infty t^{n-1+2s} K_\Phi(wt)\, T(X;t)\, dt,$$

where

(31) $$T(X;t) = \psi(X;t) - \psi(X;+0).$$

In virtue of notation (15) we have

(32) $$I = c\,a\,\psi(X;0) + I^*,$$

where

$$I^* = c\, w^{n+2s} \int_0^\infty t^{n-1+2s} K_\Phi(wt)\, T(X;t)\, dt.$$

Since $T(X;t)$ is of bounded variation in $0<t<\infty$, the left and right-hand limits $T(X;t-o)$ and $T(X;t+o)$ exist for every $t>o$. Without restricting the generality we may assume that $T(X;t)$ is bounded and monotonic because any real-valued function of bounded variation is the difference of two bounded monotonic functions. We split the integral I^* in the following way

(33) $$I^* = c\, w^{n+2s}\left(\int_0^\lambda + \int_\lambda^\mu + \int_\mu^\infty\right) t^{n-1+2s} K_\Phi(wt)\, T(X;t)\, dt = I_1^* + I_2^* + I_3^*,$$

where λ can be chosen so that $|T(X;t)|<\varepsilon$ for $0\leqslant t\leqslant\lambda$ and any arbitrary $\varepsilon>0$.

We shall estimate the last three integrals applying the meanvalue theorem to each one.

$$I_1^* = c\, w^{n+2s}\Big\{T(X;+0)\int_0^{\lambda_1} t^{n-1+2s} K_\Phi(wt)\, dt +$$

$$+ T(X;\lambda-0)\int_{\lambda_1}^{\lambda} t^{n-1+2s} K_\Phi(wt)\, dt\Big\}, \quad (0<\lambda_1<\lambda).$$

According to (31) we have $T(X;+0)=0$, and then

$$I_1^* = c\, T(X;\lambda-0) \int_{\lambda_1 w}^{\lambda w} u^{n-1+2s} K_\Phi(u)\, du.$$

Since the last integral is finite, then, with respect to the choice of λ, we get

(34) $$|I_1^*| < C_1\,\varepsilon,$$

where $\varepsilon>0$ is arbitrarily small.

Further

$$I_3^* = c\{T(X; \mu+0) \int_{\mu w}^{\mu_2 w} u^{n-1+2s} K_\Phi(u)\, du + T(X; \infty) \int_{\mu_2 w}^{\infty} u^{n-1+2s} K_\Phi(u)\, du, \quad (\mu < \mu_2 < \infty).$$

Since, because of the existence of integral (15), the last two integrals become arbitrarily small when w becomes sufficiently large, say for $w > w_0$, and $T(X; \mu+0)$ and $T(X; \infty)$ being finite, we have

$$(35) \qquad |I_3^*| < C_3 \varepsilon \quad \text{for all } w > w_0.$$

Similarly we get

$$I_2^* = c\left\{T(X; \lambda+0) \int_{\lambda w}^{\mu_1 w} u^{n-1+2s} K_\Phi(u)\, du + \right.$$

$$\left. + T(X; \mu-0) \int_{\mu_1 w}^{\mu w} u^{n-1+2s} K_\Phi(u)\, du \right\}, \quad (\lambda < \mu_1 < \mu)$$

where $T(X; \lambda+0)$ and $T(X; \mu-0)$ are finite and both integrals can be made arbitrarily small if w is sufficiently large, say $w > w_1$. Therefore we have

$$(36) \qquad |I_2^*| < C_2 \varepsilon \quad \text{for all } w > w_1.$$

Since I is equal to $\Delta_X^s [S_w^\Phi(X)]$, the result (28) of Theorem 1 follows from (32), (33), (34), (35) and (36).

3. Now we are going to prove a theorem (Theorem 2) which connects the asymptotic behaviour of the spherical means of higher order of the function $f(X)$ with the asymptotic behaviour of the Riesz mean $S_w^q(X)$ of the spherical partial sums of series (1) when the behaviour of $S_w^q(X)$ is connected with the behaviour of a slowly oscillating function. A such theorem was mentioned in [5], but without the proof.

Theorem 2. *If*

$$(37) \qquad S_w^q(X) \simeq w^p L(w) \quad \text{as } w \to \infty$$

where

$$(38) \qquad p > -2(q+1)$$

and $L(w)$ is a slowly oscillating function at infinity, then

$$f_s(X; y) \simeq \frac{2^p \Gamma(1+q+p/2)\, \Gamma(s+n/2)}{\Gamma(1+q)\, \Gamma[s+(n-p)/2]} y^{-p} L(1/y) \quad as\ y \to 0$$

for

$$(39) \qquad s > \max\{1, p+q-(n-3)/2\}.$$

Proof. We shall use relation (11) with $k = q$ if $q > (n-1)/2$, and with $k = (n-1)/2 + \varepsilon$, $\varepsilon > 0$ if $q \leqslant (n-1)/2$. In virtue of hypothesis (37) we can choose such a number δ that

$$(40) \qquad |S_w^q(X) - w^p L(w)| \leqslant \varepsilon w^p L(w)$$

for $w \geqslant \delta$ and any $\varepsilon > 0$.

We shall write (11) in the following form

$$f_s(X;y)=c_1\,y^{2q+2}\int_0^{\delta} S_w^q(X)\,w^{2q+1}\,V_{s+q+n/2}(yw)\,dw+$$

$$+c_1\,y^{2q+2}\int_{\delta}^{\infty}[S_w^q(X)-w^p L(w)]\,w^{2q+1}\,V_{s+q+n/2}(yw)\,dw+$$

$$+c_1\,y^{2q+2}\int_{\delta}^{\infty} w^{p+2q+1}\,L(w)\,V_{s+q+n/2}(yw)\,dw=$$

$$=H_1+H_2+H_3. \tag{41}$$

Now we are going to estimate the last three integrals. We shall write H_3 in the form

$$H_3=c_1\,v^p\int_{\delta/v}^{\infty} u^{p+2q+1}\,V_{s+q+n/2}(u)\,L(vu)\,du=$$

$$=c_1\,v^p\Big(\int_0^{\infty}-\int_0^{\delta/v}\Big)=H_{31}+H_{32},\ \text{where}\ v=1/y. \tag{42}$$

We can apply property (β) of the slowly oscillating function to integral H_{31} because the function

$$g(u)=u^{p+2q+1}\,V_{s+q+n/2}(u)$$

satisfies conditions (13). Namely, in virtue of (20) and (38) the first of integrals (13) exists, and according to (21) and (39) the second of integrals (13) exists too. Then with respect to (14) we obtain

$$H_{31}\cong c_1\,v^p\,L(v)\int_0^{\infty} u^{p+2q+1}\,V_{s+q+n/2}(u)\,du\quad \text{as } v\to\infty. \tag{43}$$

Since p and q satisfy conditions (38) and (39), we can use the following formulae [3]

$$\int_0^{\infty} u^{\alpha-1}\,V_\beta(u)\,du=2^{\alpha-\beta-1}\,\Gamma(\alpha/2)\,[\Gamma(1+\beta-\alpha/2)]^{-1}\ \text{for}\ 0<\alpha<\beta+3/2,$$

and then (43) becomes

$$H_{31}\cong\frac{2^p\,\Gamma(1+q+p/2)\,\Gamma(s+n/2)}{\Gamma(1+q)\,\Gamma[s+(n-p)/2]}\,y^{-p}\,L(1/y)\quad \text{as}\ y\to 0, \tag{44}$$

where $y=1/v$.

Now we shall estimate the integral H_{32}. Employing (20) we have

$$|H_{32}|\leqslant M_1\,c_1\,v^p\int_0^{\delta/v} u^{p+2q+1}\,L(vu)\,du.$$

Since the function $L(x)$ is positive and continuous in the interval $0 \leqslant x < \infty$, we get

$$|H_{32}| \leqslant M_2 c_1 v^p \int_0^{\delta/v} u^{p+2q+1} du = \frac{M_3}{v^{2q+2}} = \frac{M_3 v^p L(v)}{v^{p+2q+2} L(v)}.$$

Exploring property (α) of the slowly oscillating functions, given by (12), and condition (38), it comes out

$$H_{32} = o[v^p L(v)] \quad \text{as} \quad v \to \infty,$$

or

(45) $$H_{32} = o[y^{-p} L(1/y)] \quad \text{as} \quad y \to 0.$$

From (42), (44) and (45) we obtain

(46) $$H_3 \cong \frac{2^p \Gamma(1+q+p/2)\, \Gamma(s+n/2)}{\Gamma(1+q)\, \Gamma[s+(n-p)/2]} y^{-p} L(1/y) \quad \text{as } y \to 0.$$

We can estimate the integral H_2 by (40).

$$|H_2| \leqslant \varepsilon c_1 y^{2q+2} \int_0^\infty w^{p+2p+1} |V_{s+q+n/2}(yw)| L(w)\, dw =$$

$$= \varepsilon c_1 v^p \int_{\delta/v}^\infty u^{p+2q+1} |V_{s+q+n/2}(u)| L(vu)\, du \leqslant$$

$$\leqslant \varepsilon c_1 v^p \int_0^\infty u^{p+2q+1} |V_{s+q+n/2}(u)| L(vu)\, du,$$

where $v = 1/y$. Under the same conditions as in the case of integral H_{31} we get

$$\int_0^\infty u^{p+2q+1} |V_{s+q+n/2}(u)| L(vu)\, du \cong$$

$$\cong L(v) \int_0^\infty u^{p+2q+1} |V_{s+q+n/2}(u)|\, du \quad \text{as} \quad v \to \infty.$$

If we split the last integral in two parts, then we shall see, using relations (38), (39), (20) and (21), that it exists. Since ε can be arbitrarily small, we obtain

$$H_2 = o[v^p L(v)] \quad \text{as} \quad v \to \infty$$

or

(47) $$H_2 = o[y^{-p} L(1/y)] \quad \text{as} \quad y \to 0.$$

Finally we have to estimate the integral H_1. Since $S_w^q(X)$ is continuous in the interval $0 \leqslant w < \infty$, it follows according to (37)

$$|S_w^q(X)| \leqslant M w^p L(w),$$

and we have

$$|H_1| \leqslant c_1 M y^{2q+2} \int_0^{\delta} w^{p+2q+1} \, |V_{s+q+n/2}(yw)| \, L(w) \, dw =$$

$$= c_1 M v^p \int_0^{\delta/v} u^{p+2q+1} \, |V_{s+q+n/2}(u)| \, L(vu) \, du, \quad v = 1/y.$$

In virtue of both (20) and the fact that the function $L(x)$ is continuous in the interval $0 \leqslant x < \infty$ it comes out

$$|H_1| \leqslant M_1 v^p \int_0^{\delta/v} u^{p+2q+1} \, du = \frac{M_3 v^p}{v^{p+2q+1}} = \frac{M_3 v^p L(v)}{v^{p+2q+1} L(v)}.$$

According to (12) and (38) we get

$$H_1 = o\left[v^p L(v)\right] \quad \text{as} \quad v \to \infty,$$

or

(48) $$H_1 = o\left[y^{-p} L(1/y)\right] \quad \text{as} \quad y \to 0.$$

From (41), (46), (47) and (48) it follows

$$f(X; y) \cong \frac{2^p \Gamma(1+q+p/2) \, \Gamma(s+n/2)}{\Gamma(1+q) \, \Gamma[s+(n-p)/2]} \, y^{-p} L(1/y) \qquad \text{as } y \to 0.$$

Therefore we have proved Theorem 2.

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