PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 19 (33), 1975, pp. 101—110

ON RELATIVE DEFORMATION AND VORTICITY IN RELATIVISTIC KINEMATICS

I. Lukačević

(Communicated November 13, 1974)

We shall consider, in this paper, the relative deformation, or quasideformation, and corresponding relative vorticity, or quasivorticity, in relativistic kinematics, i.e. the deformation and the vorticity in a material continuum, with respect to the proper time of a local frame which is not comoving with it.

Relative deformation has already been investigated in relativistic mechanics. Cattaneo [7] and other authors have considered tensors which have similarity with the ones we shall analyse here. We start from two definitions of the tensor of quasideformation which represent, each one, an extension of the well-known definition of deformation, which is given as the deviation from Born's rigidity. We compare these two tensors, find their timelike eigenvectors, and obtain the condition for their mutual identity. We obtain also several cases of degeneracy (in dimensions) for them, and a limiting velocity of the continuum with respect to the observer's frame.

In the second part we introduce tensors of relative vorticity (we shall give them the name of *quasivorticity*) analogous to previous tensors of *quasideformation.* The divergence of the corresponding vorticity vector gives an expression similar to the one given by Ehlers [8] for proper (as we call it) vorticity. We form the scalar invariants for one of these pairs of tensors (deformation-vorticity) and obtain, for the vanishing of these invariants, an expression for translation analogous to the one known for proper tensors. For the second pair deformation-vorticity, these consequences are stronger, and not so obvious for interpretation.

* * *

We consider, in space-time V_4, a material continuum and two fields of velocities, which are, by hypothesis, second-order derivable. One of these is u^α. the field of velocities of the continuum, the other is ξ^α, a field of timelike unit vectors given at each point of that portion of space-time and independent of u^α. So:

$$(1.1)\qquad g_{\alpha\beta}\, u^\alpha u^\beta = g_{\alpha\beta}\, \xi^\alpha \xi^\beta = -1 \quad (\alpha, \beta = 1, 2, 3, 4)$$

We shall interprete ξ^α *locally* as the four velocity of an observer which moves independently of the continuum. We shall examine the next two questions: 1) obtain the local deformation, with respect to the observer of velocity ξ^α, of the components of the metric tensor $h_{\alpha\beta}$, spacelike with respect to u^α, and 2) write the spacelike projection, orthogonal to u^α, of the deformation of the metric tensor with respect to ξ^α. Then analyse and compare the two tensors obtained above. Let us remark that the second formulation of the tensor of relative deformation or *quasideformation*, mentioned above, is similar to that considered by several authors (cf Romano [9], Grassini]12]) and that Greenberg [11] introduces a tensor of such type, obtained by means of two mutually orthogonal vectors, one of which is timelike, instead of two timelike vectors, as in our case.

The first definition of the quasideformation is given by the Lie (or, convective) derivative of $h_{\alpha\beta}$ with respect to ξ^α:

(1.2) $$\nu_{\alpha\beta} = \mathscr{L}_\xi h_{\alpha\beta} = \mathscr{L}_\xi (g_{\alpha\beta} + u_\alpha u_\beta)$$

where $\mathscr{L}_\xi$ denotes convective derivation with respect to ξ^α.

The second definition of quasideformation is obtained by projecting on directions spacelike with respect to u^α the Lie derivative $\mathscr{L}_\xi g_{\alpha\beta}$:

(1.3) $$\tau_{\alpha\beta} = h^\gamma_\alpha h^\delta_\beta \mathscr{L}_\xi g_{\gamma\delta}$$

We make the natural assumption that u^α and ξ^α respectively are the fields of tangent unit vectors of two congruences of timelike curves.

Written explicitly, our tensors are:

(1.2′) $$\nu_{\alpha\beta} = \nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha + u_\alpha \xi^\gamma \nabla_\gamma u_\alpha + u_\beta \xi^\gamma \nabla_\gamma u_\alpha + u_\alpha u^\gamma \nabla_\beta \xi_\gamma + u_\beta u^\gamma \nabla_\alpha \xi_\gamma$$

(1.3′) $$\tau_{\alpha\beta} = \nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha + u_\alpha u^\gamma \nabla_\gamma \xi_\beta + u_\beta u^\gamma \nabla_\gamma \xi_\alpha +$$
$$+ u_\alpha u^\gamma \Delta_\beta \xi_\gamma + u_\beta u^\gamma \Delta_\alpha \xi_\gamma + 2\, u_\alpha u_\beta u^\gamma u^\delta \nabla_\gamma \xi_\delta$$

where ∇ denotes the symbol of covariant derivation.

The second of our two symmetric tensors, $\tau_{\alpha\beta}$ is, by its definition, orthogonal to u^α. It has then 6 independent components, spacelike with respect to an observer comoving with the continuum. Meanwhile the tensor $\nu_{\alpha\beta}$, when considered by the comoving, or any other observer, has timelike components; it has 10 independent components in the general case. We shall examine the relations existing between these two quasideformation tensors.

Let us consider an *arbitrary* vector h^α, limited only by the condition of being orthogonal to u^α, $h^\alpha u_\alpha = 0$. When writing this condition at every point of the domain considered, its convective derivative, with respect to any parameter in space-time, will vanish. We consider h^α as an arbitrary vector field, limited only by the preceding requirement. So:

$$u^\beta \mathscr{L}_\xi h_\beta + h_\beta \mathscr{L}_\xi u^\beta = 0.$$

Examine now what does mean the condition that one of the terms in the above formula be equal to zero

(1,4) $$u^\beta \mathscr{L}_\xi h_\beta = 0$$

or explicitly

$$\xi^\alpha u^\beta \nabla_\alpha h_\beta + u^\beta h^\alpha \nabla_\beta \xi_\alpha = 0$$

If we form the difference between tensors $\nu_{\alpha\beta}$ and $\tau_{\alpha\beta}$, it results from (1.2′) and (1.3′) that, because of the orthogonality of u^β and ξ^β:

(1.5) $$(\nu_{\alpha\beta}-\tau_{\alpha\beta})\,h^\beta=0$$

or explicitly

(1.5′) $$u_\beta\, u^\alpha\,(\xi^\gamma\nabla_\gamma h_\alpha+h^\gamma\,\nabla_\alpha\xi_\gamma)=0.$$

Hence, in the direction of a vector of the spacelike field h^α, the two tensors of relative deformation, or quasideformation, have equal components. In the case when that condition holds for every spacelike h^α (with respect to u^α). $\tau_{\alpha\beta}$ is completely identical with spacelike components of $\nu_{\alpha\beta}$.

The condition (1.4) gives, because of the relation preceding it:

(1.4′) $$h_\beta\,\mathcal{L}_\xi\, u^\beta=0.$$

We shall write, in order to express the fact that for the vector h^β, relation (1.4′) holds, that the Lie derivative of u^α with respect to ξ^α is proportional to u^α:

(1.4″) $$\mathcal{L}_\xi\, u^\alpha=\varphi\, u^\alpha.$$

When unit vectors u^α and ξ^α are forming a two-surface, the Lie derivative of one of them with respect to the other (if taking contravariant coordinates only) is equal to their arbitrary linear combination (cf [11]). Relation (1.4″) is sufficient to make these vectors surface-forming. After an examination of preceding relations, we shall establish that the necessary and sufficient condition for the complete identity of $\nu_{\alpha\beta}$ with $\tau_{\alpha\beta}$ is expressed by (1.4″). Because of:

$$u^\alpha\,\mathcal{L}_\xi\, u_\alpha+u_\alpha\,\mathcal{L}_\xi\, u^\alpha=0$$

we shall have from (1.4″):

(1.6) $$\varphi=u^\alpha\, u^\beta\,\nabla_\alpha\xi_\beta$$

and from (1.2′) we obtain

$$\nu_{\alpha\beta}\, u^\beta=-g_{\alpha\beta}\,\mathcal{L}_\xi\, u^\beta+u_\alpha\, u^\beta\, u^\gamma\,\nabla_\beta\xi_\gamma=(u^\beta\, u^\gamma\,\nabla_\beta\xi_\gamma-\varphi)\, u_\alpha$$

We draw therefrom the conclusion that u_α can be an eigenvector of $\nu_{\alpha\beta}$ only trivially, with a null eigenvalue. In that case $\tau_{\alpha\beta}$ is identical with $\nu_{\alpha\beta}$.

Let us examine the relations between $\nu_{\alpha\beta}$ and the deformation tensor $\sigma_{\alpha\beta}$, vanishing in the case of Born's rigidity (cf Ehlers [8], Synge [3] Rayner [5]). Tensor $\sigma_{\alpha\beta}$ is defined by the expression:

(1.7) $$\sigma_{\alpha\beta}=\mathcal{L}_u\, h_{\alpha\beta}=\nabla_\alpha u_\beta+\nabla_\beta u_\alpha+u_\alpha w_\beta+u_\beta w_\alpha\qquad (w_\alpha=u^\gamma\,\nabla_\gamma u_\alpha)$$

Remark that $\sigma_{\alpha\beta}$ has, by Ehlers' definition, an additional term chosen in order to make the trace σ^α_α vanish. It is then a tensor of „relative deformation" in the sense of classical mechanics. We shall write these tensors with additional terms corresponding to our case:

(1.2″) $$\tilde{\nu}_{\alpha\beta}=\nu_{\alpha\beta}-\frac{1}{4}\,\nu^\gamma_\gamma\, g_{\alpha\beta}$$

(1.3″) $$\tilde{\tau}_{\alpha\beta}=\tau_{\alpha\beta}-\frac{1}{3}\,\tilde{\tau}^\gamma_\gamma\, h_{\alpha\beta}$$

$\tilde{\tau}_{\alpha\beta}$ remains a tensor of the 3-space orthogonal to u^α. In fact the difference between $\nu_{\alpha\beta}$, $\tau_{\alpha\beta}$ and $\tilde{\nu}_{\alpha\beta}$, $\tilde{\tau}_{\alpha\beta}$ respectively is not essential for us.

We shall apply to the tensor $h_{\alpha\beta}$, an identity holding for second order convectives derivatives (cf Yano [4]):

$$\mathscr{L}_u \mathscr{L}_\xi h_{\alpha\beta} - \mathscr{L}_\xi \mathscr{L}_u h_{\alpha\beta} = \mathscr{L}_\lambda h_{\alpha\beta} \tag{1.8}$$

where $\lambda^\alpha = \mathscr{L}_u \xi^\alpha = -\mathscr{L}_\xi u^\alpha$.

By (1.2) and (1.7) preceding relation will read:

$$\mathscr{L}_u \nu_{\alpha\beta} - \mathscr{L}_\xi \sigma_{\alpha\beta} = \mathscr{L}_\lambda h_{\alpha\beta}. \tag{1.8'}$$

If applying this formula to the case when (1.4″) holds, i.e. when $\tau_{\alpha\beta}$ is identical with $\nu_{\alpha\beta}$, we shall have:

$$\mathscr{L}_u \nu_{\alpha\beta} - \mathscr{L}_\xi \sigma_{\alpha\beta} = \mathscr{L}_c h_{\alpha\beta} \tag{1.9}$$

where $\mathscr{C}^\alpha = -\varphi u^\alpha$, as it is simple to verify, then

$$\mathscr{L}_u \nu_{\alpha\beta} - \mathscr{L}_\xi \sigma_{\alpha\beta} = -\varphi \sigma_{\alpha\beta}. \tag{1.10}$$

We have so by (1.8) the equivalence of the conditions (1.10) and (1.4″), which is itself equivalent to (1.5). Hence:

$$\nu_{\alpha\beta} = \tau_{\alpha\beta} \Leftrightarrow \mathscr{L}_u \nu_{\alpha\beta} - \mathscr{L}_\xi \sigma_{\alpha\beta} = -\varphi \sigma_{\alpha\beta}.$$

If the difference between convective derivatives of tensors $\sigma_{\alpha\beta}$ and $\tau_{\alpha\beta}$, with respect to conjugate world lines having u^α and ξ^α as tangent unit vectors, is proportional to $\sigma_{\alpha\beta}$, tensors $\nu_{\alpha\beta}$ and $\tau_{\alpha\beta}$ are equal.

The relation (1.10) expresses the fact that $\nu_{\alpha\beta}$ has no components in the direction of u^α.

We point out the fact that $\tau_{\alpha\beta}$ represents the spacelike projection of $\nu_{\alpha\beta}$ with respect to u^α:

$$\nu_{\gamma\delta} h^\gamma_\alpha h^\delta_\beta = h^\gamma_\alpha h^\delta_\beta \mathscr{L}_\xi h_{\gamma\delta} = h^\gamma_\alpha h^\delta_\beta \mathscr{L}_\xi g_{\gamma\delta} = \tau_{\alpha\beta}$$

This justifies the term of total quasideformation, given to $\nu_{\alpha\beta}$, at the difference of $\tau_{\alpha\beta}$, which is only its orthogonal projection.

Let us remark that the tensor of "unstationarity", defined by

$$\mathscr{L}_u g_{\alpha\beta} = \nabla_\alpha u_\beta + \nabla_\beta u_\alpha$$

can have u^α as an eigenvector only in the case of vanishing acceleration, and then corresponding eigenvalue is zero. Deformation tensor $\sigma_{\alpha\beta}$ in (1.7), by its definition, has no components in the direction of u_α. Remark also that if $\nu_{\alpha\beta}$ reduces to $\tau_{\alpha\beta}$ in some domain in which considered continuum has Born's rigidity, it does not depend locally on the proper time of that medium. In the same way, if a family of timelike world lines, given by ξ^α exists, such that $h_{\alpha\beta}$ does not depend locally on their proper time, it results that the convective derivative of $\sigma_{\alpha\beta}$ with respect to that parameter is proportional to $\sigma_{\alpha\beta}$ itself. But the inverse does not hold.

From the preceding we have always

$$\nu_{\alpha\beta} u^\alpha u^\beta = 0 \tag{1.11}$$

which is easy to verify. As (1.4″) can be written

$$\mathcal{L}_u \xi^\alpha = -\varphi u^\alpha$$

we obtain, putting the value of φ:

(1.12) $$u^\beta u^\gamma \nabla_\beta \xi_\gamma - (u^\alpha \xi_\alpha)^{-1} \xi^\beta \xi^\gamma \nabla_\beta u_\gamma = 0.$$

Let us examine now the conditions for ξ^α to be an eigenvector of $\nu_{\alpha\beta}$:

$$\nu_{\alpha\beta} \xi^\beta = \mathcal{D}_\xi \xi_\alpha + \mathcal{L}_\xi (\xi^\beta u_\beta u_\alpha) = \chi \xi_\alpha$$

where $\mathcal{D}_\xi$ is the symbol of absolute derivation in the direction of ξ^β. Then:

(1.13) $$\chi = -2 \vartheta \xi^\beta \partial_\alpha \vartheta$$

where we have put $\vartheta = \xi^\beta u_\beta$. When substituting that eigenvalue in the preceding relation:

(1.14) $$\nu_{\alpha\beta} \xi^\beta = -2 \vartheta \xi^\beta \partial_\beta \vartheta \cdot \xi_\alpha$$

It yields from (1.13) that if the scalar product of u^α with ξ^α remains constant along world lines of ξ^α, the corresponding eigenvalue χ vanishes. Let us find the eigenvectors of $\nu_{\alpha\beta}$ in the 2-plane u^α, ξ^α under the assumption that θ remains constant along ξ^α world lines. The right hand side in (1.14) remains then equal to zero. We have so:

(1.15) $$\begin{aligned} \nu_{\alpha\beta} (a u^\beta + b \xi^\beta) &= \varkappa \ (a u_\alpha + b \xi_\alpha) \\ \nu_{\alpha\beta} (a' u^\beta + b' \xi^\beta) &= \varkappa' \ (a' u_\alpha + b' \xi_\alpha) \end{aligned}$$

$\nu_{\alpha\beta}$ being symmetric, its eigenvalues are in the general case distinct, with mutually orthogonal eigenvectors (cf Synge [3]). So different vectors which correspond in (1.15) to different eigenvalues $\varkappa$ and $\varkappa'$, must satisfy the orthogonality condition:

(1.16) $$aa' + bb' - (a'b + ab') \vartheta = 0.$$

The two relations (1.15) are equivalent to the next two:

(1.15′) $$\begin{aligned} c\, \nu_{\alpha\beta} \xi^\beta &= (\varkappa - \varkappa')(a u_\alpha + c \xi_\alpha), \quad c = b - \frac{1}{a'} ab' \\ d\, \nu_{\alpha\beta} u^\beta &= (\varkappa - \varkappa')(d u_\alpha + b \xi_\alpha), \quad d = a - \frac{1}{b'} a'b. \end{aligned}$$

Because of:

(1.17) $$\xi^\alpha \partial_\alpha \vartheta = 0 \ \left(\text{we shall write } \xi^\alpha \partial_\alpha \equiv \frac{d}{d\tau}\right)$$

We have (1.11) and one relation more:

$$\nu_{\alpha\beta} u^\alpha u^\beta = \nu_{\alpha\beta} \xi^\alpha \xi^\beta = 0.$$

So that we obtain from (1.15′):

$$(\varkappa - \varkappa')(a \vartheta - c) = 0, \quad (\varkappa - \varkappa')(d - b \vartheta) = 0.$$

There are two possibilities now:

$$\text{a) } \varkappa = \varkappa' \quad \text{b) } \vartheta = \frac{c}{a} = \frac{d}{b}$$

For a) we have two distinct eigenvectors corresponding to one eigenvalue, which means that χ is a double root at least. But u^α and ξ^α being then eigenvectors, and their eigenvalues being equal to zero, as we have established, $\varkappa$ is also zero. In that case $\nu_{\alpha\beta}$ has no components in the local 2-plane u^α, ξ^α. For b) we have:

$$\vartheta = \frac{b}{a} - \frac{b'}{a'} = \frac{a}{b} - \frac{a'}{b'} \Rightarrow aa' = -bb'$$

Then (1.16) reduces to

$$a'b + ab' = 0$$

From the above relations it yields:

$$a = \begin{cases} b \Rightarrow a' = -b' \\ -b \Rightarrow a' = b' \end{cases}$$

Therefrom $\vartheta = \pm 2$.

In the special case when the scalar product of u^α and ξ^α is equal to ± 2 (which is possible, the metric being indefinite) the eigenvectors of $\nu_{\alpha\beta}$ can exist in the plane u^α, ξ^α. These two vectors being oriented towards future, it is easy to verify that only the value $\vartheta = -2$ is correct. The eigenvectors of $\nu_{\alpha\beta}$ are then:

$$\lambda_\alpha = u_\alpha - \xi_\alpha, \quad \mu_\alpha = u_\alpha + \xi_\alpha$$

the first of them being spacelike, the second one timelike.

Hence, in the whole of the domains

$$\text{(1.18)} \qquad -\infty < \vartheta < -2 \text{ and } -2 < \vartheta < -1$$

for $\dfrac{d\vartheta}{d\tau} = 0$ *the tensor of total quasideformation* $\nu_{\alpha\beta}$ *(and also* $\tau_{\alpha\beta}$*) cannot have extremal values in the 2-plane* ξ^α, u^α. *It reduces then to a two dimensional tensor in the local 2-plane which is the orthogonal complement of* u^α, ξ^α.

All the eigenvectors of a symmetric tensor of rank 2 being mutually orthogonal in space-time, it results that if $\nu_{\alpha\beta}$ has two extremal values in those directions which are purely spacelike for both the medium with four velocity u^α and the observer ξ^α, subjected to the condition (1.18), which makes the intensity of the three-velocity of the medium steady with respect to the observer's proper time, it becomes a two dimensional tensor.

The motion of the continuum observed from any „rest" frame ξ^α, corresponding to $\vartheta = -2$, gives a three velocity equal to $\frac{1}{2} c\sqrt{3}$, in other words a very high one for any continuum. We can consider that velocity as a limiting one, despite the fact that it was obtained from purely kinematical considerations.

Let us return to the tensor of „proper deformation" $\sigma_{\alpha\beta}$, given by (1.7). It is simple to obtain the next quadratic form:

$$\text{(1.19)} \qquad \sigma_{\alpha\beta}\xi^\alpha\xi^\beta = 2\,(\xi^\alpha\xi^\beta\nabla_\alpha u_\beta - \vartheta\, u^\alpha u^\beta\,\nabla_\alpha\xi_\beta) + 2\,\vartheta\, u^\alpha\partial_\alpha\vartheta.$$

In the case when $\nu_{\alpha\beta}$ reduces to $\tau_{\alpha\beta}$ we obtain, in virtue of (1.12), that the first term in the right hand side of (1.19) vanishes. Furthermore, if ϑ remains constant along the world lines of the fluid stream u^α, the above quadratic form becomes equal to zero. On the other hand, $\sigma_{\alpha\beta}$, being a spacelike tensor (it belongs to the definite part of space-time), and expressing deformation, must have positive coefficients written in its principal frame (the „deformation ellipse"). The nullity of the quadratic form in (1.19) means then that $\sigma_{\alpha\beta}$ either vanishes or has no components in the direction of ξ^β. So with the condition (1.12) and:

$$u^\alpha \partial_\alpha \vartheta = 0 \tag{1.20}$$

$\sigma_{\alpha\beta}$ becomes a two dimensional tensor, with no components in the plane u^α, ξ^α.

The independence of ϑ on any parameter in the two-surface u_α, ξ_α, given by eqs. (1.17) and (1.20), joined to the condition on $\nu_{\alpha\beta}$ to have extremal values in the orthogonal complement of that plane (or to have invariable extremal values in both the rest and the comoving frames) has, in virtue of (1.4″) the consequence that world lines are surface forming as it was already pointed out. We have at each point of those surfaces completely isogonal pairs of world lines determined by their tangent vectors. $\nu_{\alpha\beta}$ and $\sigma_{\alpha\beta}$ are then two dimensional and orthogonal with respect to above local 2-planes. There is a two-parameter family of such surfaces, the pseudo-angles between local basic vectors on them varying from one to other, up to the value $g_{\alpha\beta} u^\alpha \xi^\beta = -2$.

In the case of given deformation tensors with above proprieties all the consequences are obvious.

* * *

We shall complete the tensors of quasideformation, considered in the previous section, introducing two skew-symmetric tensors complementary, by their definition, to previous ones, and form correspondig invariants.

We introduce first:

$$\Omega_{\alpha\beta} = \nabla_\alpha \xi_\beta - \nabla_\beta \xi_\alpha + u_\alpha \xi^\gamma \nabla_\gamma u_\beta - u_\alpha u^\gamma \nabla_\alpha \xi_\gamma + u_\beta u^\gamma \nabla_\alpha \xi_\gamma - u_\beta \xi^\gamma \nabla_\gamma u_\alpha \tag{2.1}$$

which is skew-symmetric and corresponding to $\nu_{\alpha\beta}$. The tensor $\Omega_{\alpha\beta}$ has, in the general case, 6 independent components.

Second we introduce the tensor:

$$\begin{aligned}\Phi_{\alpha\beta} &= \Omega_{\gamma\delta} h^\gamma_\alpha h^\delta_\beta = (\nabla_\gamma \xi_\delta - \nabla_\delta \xi_\gamma) h^\gamma_\alpha h^\delta_\beta = \\ &= \nabla_\alpha \xi_\beta - \nabla_\beta \xi_\alpha + u_\alpha u^\gamma \nabla_\gamma \xi_\beta - u_\alpha u^\gamma \nabla_\beta \xi_\gamma + u_\beta u^\gamma \nabla_\alpha \xi_\gamma - u_\beta u^\gamma \Delta_\gamma \xi_\alpha\end{aligned} \tag{2.2}$$

corresponding to $\tau_{\alpha\beta}$ in the same way as $\Omega_{\alpha\beta}$ to $\nu_{\alpha\beta}$, which is obvious from the above relation. We can verify that the condition:

$$(\Omega_{\alpha\beta} - \Phi_{\alpha\beta}) h^\beta = 0 \quad (h^\alpha u_\alpha = 0) \tag{2.3}$$

which holds when spacelike components of $\Omega_{\alpha\beta}$ and $\Phi_{\alpha\beta}$ coincide, has also the consequence that $\Omega_{\alpha\beta}$ has no components in the direction of u_α, and coincides completely with $\Phi_{\alpha\beta}$. That fact is expressed by (1.4″), which results from (2.3).

We point out the fact that under the condition (1.17) $\Omega_{\alpha\beta}$, at the difference of $\nu_{\alpha\beta}$, is not less in dimension. The same conclusion holds under (1.20).

Let us return to the more general case of a four dimensional tensor of *total vorticity* $\Omega_{\alpha\beta}$. If we introduce a vorticity vector ψ^{α}:

(2.4) $$\psi^{\alpha} = \frac{1}{2} \varepsilon^{\alpha\beta\gamma\delta} u_{\beta} \Omega_{\gamma\delta}$$

in analogy with vectors used (cf Lichnerowicz [2], Synge [6], Ehlers [8], and others) which are tangent vectors of vorticity world lines, we shall have for an irrotational motion:

(2.5) $$\psi^{\alpha} = 0 \Rightarrow u_{\beta} \Omega_{\gamma\delta} + u_{\gamma} \Omega_{\delta\beta} + u_{\delta} \Omega_{\beta\gamma} = 0$$

Here the world lines are, for rotational motion, spacelike, ψ^{α} being orthogonal to u_{α}. In every case the so-called vectors of kinematical vorticity are spacelike. If we introduce an auxiliary vector $\vartheta_{\gamma} = \Omega_{\beta\gamma} u^{\beta}$, we obtain after scalar multiplying of (2.5) by u^{α} that:

(2.5′) $$\Omega'_{\delta\gamma} = \Omega_{\delta\gamma} + \vartheta_{\gamma} u_{\delta} - \vartheta_{\delta} u_{\gamma} = 0$$

the skew-symmetric tensor $\Omega'_{\gamma\delta}$ vanishes as the consequence of the vanishing of ψ^{α} and vice versa. In the case when (2.3) holds all the preceding discussion would lead to the conclusion that $\vartheta^{\alpha} = 0$. Therefrom (2.5′) would yield that $\Omega_{\gamma\delta}$, like $\Phi_{\gamma\delta}$, must vanish.

We shall examine the question of the divergence of ψ^{α}. We have first

$$\nabla_{\alpha} \psi^{\alpha} = \varepsilon^{\alpha\beta\gamma\delta} \nabla_{\alpha} (u_{\beta} \nabla_{\gamma} \xi_{\delta}) = \varepsilon^{\alpha\beta\gamma\delta} \nabla_{\alpha} u_{\beta} \cdot \nabla_{\gamma} \xi_{\delta} - $$
$$- \frac{1}{2} \varepsilon^{\alpha\beta\gamma\delta} u_{\alpha} (\nabla_{\beta} \nabla_{\gamma} \xi_{\delta} - \nabla_{\gamma} \nabla_{\beta} \xi_{\delta}).$$

The second term of the right hand side can be, with the help of Ricci's identity, written in the form

$$\nabla_{\alpha} \psi^{\alpha} = \varepsilon^{\alpha\beta\gamma\delta} \nabla_{\alpha} u_{\beta} \cdot \nabla_{\gamma} \xi_{\delta} - \frac{1}{2} \varepsilon^{\alpha\beta\gamma\delta} u_{\alpha} \mathcal{R}_{\zeta\delta\beta\gamma} \xi^{\zeta}$$

or finally:

(2.6) $$\nabla_{\alpha} \psi^{\alpha} = \tilde{\psi}_{\alpha\beta} (* \psi^{\alpha\beta})$$

because the second term in right hand side of the previous relation has vanished in virtue of the algebraic identities satisfied by the Riemann-Christoffel tensor $\mathcal{R}_{\alpha\beta\gamma\delta}$. The terms $\tilde{\psi}_{\alpha\beta}$ and $*\psi_{\alpha\beta}$ are respectively:

$$\tilde{\psi}_{\alpha\beta} = \frac{1}{2} (\nabla_{\alpha} \xi_{\beta} - \nabla_{\beta} \xi_{\alpha}), \quad * \psi^{\alpha\beta} = \frac{1}{2} \varepsilon^{\alpha\beta\gamma\delta} (\nabla_{\gamma} u_{\delta} - \nabla_{\delta} u_{\gamma}).$$

The divergence in (2.6) becomes (cf Ehlers [8]), in the case of proper vorticity tensor, equal to the product $v_{\alpha} w^{\alpha}$ $(w^{\alpha} = u^{\beta} \nabla_{\beta} u_{\alpha})$, v^{α} being the vorticity vector. This can be verified in (Lichnerowicz [2]); our scalar product of forms $\tilde{\psi}$ and $*\psi$ reduces to that expression for $u^{\alpha} = \xi^{\alpha}$.

Let us form now the invariants for the pair of tensors $\tau_{\alpha\beta}$ and $\Phi_{\alpha\beta}$. It is simpler that for the other pair of tensors. We have first:

$$\tau_{\alpha\beta}+\Phi_{\alpha\beta}=2\,(\nabla_\alpha \xi_\beta+u_\alpha u^\gamma \nabla_\gamma \xi_\beta+u_\beta u^\gamma \nabla_\alpha \xi_\gamma+u_\alpha u_\beta u^\gamma u^\delta \nabla_\gamma \xi_\delta) \tag{2.7}$$

Using next symbols:

$$\tilde{w}_\beta=u^\gamma \nabla_\gamma \xi_\beta; \quad \tau^2=\tau_{\alpha\beta}\tau^{\alpha\beta}; \quad \Phi^2=\Phi_{\alpha\beta}\Phi^{\alpha\beta}$$

we shall have

$$\tau^2+\Phi^2=4\,[g^{\alpha\beta}(g^{\gamma\delta}+u^\gamma u^\delta)\nabla_\alpha \xi_\gamma \cdot \nabla_\beta \xi_\delta+\tilde{w}_\alpha \tilde{w}^\alpha+(u_\alpha \tilde{w}^\alpha)^2]. \tag{2.8}$$

The fundamental scalar invariants τ^2 and Φ^2 are (cf Ehlers [8]) are strictly positive.

In the case when (2.8), i.e. the scalar invariants Φ^2 and τ^2, are equal to zero, we have accordingly the consequence that tensors $\tau_{\alpha\beta}$ and $\Phi_{\alpha\beta}$ are equal to zero. After writing that, and multiplying (1.3′) and (2.2) by ξ^α (having in mind its orthogonality to $\tilde{w}_\alpha$), we obtain combining:

$$u^\gamma \nabla_\beta \xi_\gamma+u^\gamma \tilde{w}_\gamma u_\beta=0. \tag{2.9}$$

Putting this in (2.6) it yields:

$$\nabla_\alpha \xi_\beta+u_\alpha \tilde{w}_\beta=0. \tag{2.10}$$

Contracting the above equation:

$$\nabla_\alpha \xi^\alpha+u^\alpha u^\beta \nabla_\alpha \xi_\beta=0.$$

This is the trace of both tensors $\tau_{\alpha\beta}$ and $\nu_{\alpha\beta}$. It expresses the absence of dilatation, with respect to u_α, of the elementary three-surface locally orthogonal to the "world-tube" of lines with tangent vector ξ^α (cf Synge [3], Greenberg [10]). The relation (2.10) represents the generalisation, in some way, of translation, in other words a "quasitranslation", and it is similar to expressions given by some authors (cf Synge [3], Salzman & Taub [1]). In fact, in the absence of rotation and deformation in the sense considered here, it is the only possibility.

Let us consider now tensors $\nu_{\alpha\beta}$ and $\Omega_{\alpha\beta}$. When combining them:

$$\begin{aligned}\nu_{\alpha\beta}+\Omega_{\alpha\beta}&=2\,(\nabla_\alpha \xi_\beta+u_\alpha \xi^\gamma \nabla_\gamma u_\beta+u_\beta u^\gamma \nabla_\alpha \xi_\gamma)\\ \nu_{\alpha\beta}-\Omega_{\alpha\beta}&=2\,(\nabla_\beta \xi_\alpha+u_\beta \xi^\gamma \nabla_\gamma u_\alpha+u_\alpha u^\gamma \nabla_\beta \xi_\gamma)\end{aligned} \tag{2.11}$$

we obtain non symmetric tensors. After multiplying by u^β:

$$\begin{aligned}(\nu_{\alpha\beta}+\Omega_{\alpha\beta})\,u^\beta&=(\nu_{\beta\alpha}-\Omega_{\beta\alpha})\,u^\beta=0\\ (\nu_{\beta\alpha}+\Omega_{\beta\alpha})\,u^\beta&=(\nu_{\alpha\beta}-\Omega_{\alpha\beta})\,u^\beta=2\,g_{\alpha\beta}(\mathcal{L}_u \xi^\beta+\varphi u^\beta)\end{aligned} \tag{2.12}$$

and by ξ^β:

$$\begin{aligned}(\nu_{\alpha\beta}+\Omega_{\alpha\beta})\,\xi^\beta&=(\nu_{\beta\alpha}-\Omega_{\beta\alpha})\,\xi^\beta=2\,(\xi^\beta \xi^\gamma \nabla_\beta u_\gamma \cdot u_\alpha+\vartheta u^\beta \nabla_\alpha \xi_\beta)\\ (\nu_{\beta\alpha}+\Omega_{\beta\alpha})\,\xi^\beta&=(\nu_{\alpha\beta}-\Omega_{\alpha\beta})\,\xi^\beta=2\,(\mathcal{D}_\xi \xi_\alpha+\vartheta \xi^\beta \nabla_\beta u_\alpha+u^\beta \mathcal{D}_\xi \xi_\beta \cdot u_\alpha).\end{aligned} \tag{2.13}$$

The first relation (2.11) yields:

$$\mathrm{Det}\,\|\nu_{\alpha\beta}+\Omega_{\alpha\beta}\|=\mathrm{Det}\,\|\nu_{\alpha\beta}-\Omega_{\alpha\beta}\|=0$$

So either strokes or columns in these tensors are not independent, but we must have in mind that tensors $\nu_{\alpha\beta}\pm\Omega_{\alpha\beta}$ are non degenerate and have, in the general case, components in all the directions of space-time. The first relation (2.13) gives us just the expression to which $\Omega_{\alpha\beta}\xi^{\beta}$ reduces when taking in account conditions (1.17) and (1.20).

If making, finally, invariants ν^2 and Ω^2 equal to zero, one make $\nu_{\alpha\beta}$ and $\Omega_{\alpha\beta}$ also vanish. Due to the fact that $\tau_{\alpha\beta}$ and $\Phi_{\alpha\beta}$ are then necessarily equal to zero, we should obtain the quasitranslation given by (2.10), but relations (2.11), being also equal to zero, have even stronger consequences.

REFERENCES

[1] Salzman, Taub, Physical Review, 95, p. 1659 (1954).

[2] A. Lichnerowicz, *Théories relativistes de la gravitation et de l'électromagnétisme*, Masson (1955).

[3] Synge, *Relativity, The Special Theory*, North-Holland (1956).

[4] K. Yano, *The theory of Lie Derivatives*, North-Holland (1957).

[5] C. B. Rayner, Atti Lincei 26, p. 478—483 (1959).

[6] Synge, *Relativity, The General Theory*, North-Holland (1960).

[7] C. Cattaneo, Cours professé au Collège de France, (1961—1962).

[8] J. Ehlers, Akad. Wiss. Abh. Math. Naturwiss., no 11 (1961).

[9] Romano, Ricerche Matematiche, Napoli, XIX, p 208—224 (1970).

[10] Ph. Greenberg Jr. Math. Anal. and Appl., 29, p. 647—667 (1970).

[11] Ph. Greenberg Jr. Math. Anal. and Appl., 30, p. 128—143 (1970).

[12] R. Grassini, Ricerche Matematiche, Napoli, XX, p. 241—252 (1971).

[13] G. Maugin, C. R. Acad. Sc., t A 65—69 (1971).

[14] Glass, Winicour, Jr. Math. Phys., Vol. no 9 (1973).

[15] I. Lukačević, 12th Yugoslav Congress for Rat. and Appl. Mechanics, Ohrid. June 1974; in printing in the „Theoretical and Applied Mechanics", Belgrade (1975).