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APPROXIMATION PROPERTIES OF A SEQUENCE OF LINEAR POSITIVE OPERATORS

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This paper is dedicated to the memory of our Professor T. Popoviciu

1. Introduction.

An example of a sequence of linear polynomial operators, with „good" approximation properties, is given by the Bernstein operators [1, 5, 10, 11 13]. These properties, for instance the convexity preservation, enable us to use the Bernstein operators in various fields of mathematics as for example Summability, Statistics [14].

The aim of this note is to introduce a sequence of linear polynomial operators and at the same time to study their properties which are similar to those of Bernstein operators.

For a fixed number ε on the interval $\left(0, \frac{1}{2}\right)$ let us write

$$K_1 := [\varepsilon, 1-\varepsilon] \subset K, \qquad K := [0,1],$$

$$\mathbf{N}_\varepsilon := \left\{\left[\frac{1}{2\varepsilon}\right], \left[\frac{1}{2\varepsilon}\right]+1, \left[\frac{1}{2\varepsilon}\right]+2, \dots\right\},$$

$$\|\cdot\|_1 := \max_{K_1} |\cdot|, \qquad \|\cdot\| := \max_K |\cdot|.$$

We define the sequence of operators $L_n : C(K) \to C(K_1)$, $n \in \mathbf{N}_\varepsilon$, by

$$L_n f := \sum_{k=0}^{n} l_{k,n} \int_0^1 \left[\frac{t}{n+1}, \frac{t+1}{n+1}, \dots, \frac{t+k}{n+1}; f\right] dt \tag{1}$$

where

$$l_{k,n}(x) := \frac{n!}{n^k (n-k)!}\left(x - \frac{1}{2n+2}\right)^k, \ n \in \mathbf{N}_\varepsilon, \ k = 0, 1, \dots, n$$

and the symbol $[x_0, x_1, \dots, x_m; \cdot]$ denotes the divided difference at the prescribed knots x_i, $i = 0, \dots, m$.

If

$$P_{n,k}(x) := \frac{(n+1)^{n+1}}{n^n}\binom{n}{k}\left(x - \frac{1}{2n+2}\right)^k \left(\frac{2n+1}{2n+2} - x\right)^{n-k},$$

$$n \in \mathbf{N}_\varepsilon, \quad k = 0, 1, \dots, n,$$

then we prove

Theorem 1. *The linear operator* $L_n: C(K) \to C(K_1)$ *(see (1)) may be defined by*

$$L_n f = \sum_{k=0}^{n} P_{n,k} \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)\,dt = \int_0^1 P_{n,\,[(n+1)t]}(x) f(t)\,dt. \tag{2}$$

Therefore L_n, $n \geqq N_\varepsilon$ *are linear positive operators.*

Proof. From (2) one proves by induction that for derivation of L_n we have

$$(L_n f)^{(j)}(x) = \frac{(n+1)^{n-j+1}\, n!\, j!}{n^n (n-j)!} \sum_{k=0}^{n-j} C_{k,j}(x) \cdot \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} \left[t,\, t+\frac{1}{n+1},\, \ldots,\, t+\frac{j}{n+1};\, f\right] dt \tag{3}$$

where

$$C_{k,j}(x) := \binom{n-j}{k} \left(x - \frac{1}{2n+2}\right)^{k} \left(\frac{2n+1}{2n+2} - x\right)^{n-j-k}.$$

Indeed,

$$(L_n f)'(x) = \frac{n^{n+1}}{n^n} \left\{ \sum_{k=1}^{n} k \binom{n}{k} \left(x - \frac{1}{2n+2}\right)^{k-1} \left(\frac{2n+1}{2n+2} - x\right)^{n-k} \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)\,dt - \right.$$

$$\left. - \sum_{k=0}^{n-1} (n-k) \binom{n}{k} \left(x - \frac{1}{2n+2}\right)^{k} \left(\frac{2n+1}{2n+2} - x\right)^{n-1-k} \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)\,dt \right\}$$

and

$$\sum_{k=1}^{n} k \binom{n}{k} \left(x - \frac{1}{2n+2}\right)^{k-1} \left(\frac{2n+1}{2n+2} - x\right)^{n-k} \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)\,dt =$$

$$= n \sum_{k=0}^{n-1} \binom{n-1}{k} \left(x - \frac{1}{2n+2}\right)^{k} \left(\frac{2n+1}{2n+2} - x\right)^{n-1-k} \int_{\frac{k+1}{n+1}}^{\frac{k+2}{n+1}} f(t)\,dt =$$

$$= n \sum_{k=0}^{n-1} \binom{n-1}{k} \left(x - \frac{1}{2n+2}\right)^{k} \left(\frac{2n+1}{2n+2} - x\right)^{n-1-k} \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f\left(t + \frac{1}{n+1}\right) dt,$$

$$\sum_{k=0}^{n-1}(n-k)\binom{n}{k}\left(x-\frac{1}{2n+2}\right)^k\left(\frac{2n+1}{2n+2}-x\right)^{n-1-k}\int\limits_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)\,\mathrm{dt}=$$

$$=n\sum_{k=0}^{n-1}\binom{n-1}{k}\left(x-\frac{1}{2n+2}\right)^k\left(\frac{2n+1}{2n+2}-x\right)^{n-1-k}\int\limits_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)\,\mathrm{dt},$$

because $(n-k)=n\binom{n-1}{k}$. It follows

$$(L_nf)'(x)=\frac{(n+1)^{n+1}}{n^{n-1}}\sum_{k=0}^{n-1}C_{k,1}(x)\int\limits_{\frac{k}{n+1}}^{\frac{k+1}{n+1}}\left[t,\,t+\frac{1}{n+1};\,f\right]\mathrm{dt}$$

and formula (3) for $j=1$ is valid. Let us assume that (3) holds true for $j=p+1$. Then we may write

$$(L_nf)^{(p+2)}(x)=\frac{(n+1)^{n-p}\,n!\,(p+1)!}{n^n\,(n-p-1)!}(X-Y)$$

where

$$X:=\sum_{k=1}^{n-p-1}k\binom{n-p-1}{k}\left(x-\frac{1}{2n+2}\right)^{k-1}\left(\frac{2n+1}{2n+2}-x\right)^{n-p-1-k}\cdot$$

$$\cdot\int\limits_{\frac{n+1}{k}}^{\frac{k+1}{n+1}}\left[t,\,t+\frac{1}{n+1},\,\ldots,\,t+\frac{p+1}{n+1};\,f\right]\mathrm{dt}$$

and

$$Y:=\sum_{k=0}^{n-p-2}(n-p-1-k)\binom{n-p-1}{k}\left(x-\frac{1}{2n+2}\right)^{k}\left(\frac{2n+1}{2n+2}-x\right)^{n-p-2-k}\cdot$$

$$\cdot\int\limits_{\frac{k}{n+1}}^{\frac{k+1}{n+1}}\left[t,\,t+\frac{1}{n+1},\,\ldots,\,t+\frac{p+1}{n+1};\,f\right]\mathrm{dt}.$$

Further, since

$$\int\limits_{\frac{k+1}{n+1}}^{\frac{k+2}{n+1}}\left[t,\,t+\frac{1}{n+1},\,\ldots,\,t+\frac{p+1}{n+1};\,f\right]\mathrm{dt}=$$

$$=\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}}\left[t,\,t+\frac{1}{n+1},\,\ldots,\,t+\frac{p+1}{n+1};\,f\left(\cdot+\frac{1}{n+1}\right)\right]dt$$

and

$$(n-p-1-k)\binom{n-p-1}{k}=(n-p-1)\binom{n-p-2}{k},$$

we may write

$$X=(n-p-1)\sum_{k=0}^{n-p-2}C_{k,p+2}(x)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}}\left[t,\,t+\frac{1}{n+1},\,\ldots,\,t+\frac{p+1}{n+1};\,f\left(\cdot+\frac{1}{n+1}\right)\right]dt$$

and

$$Y=(n-p-1)\sum_{k=0}^{n-p-2}C_{k,p+2}(x)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}}\left[t,\,t+\frac{1}{n+1},\,\ldots,\,t+\frac{p+1}{n+1};\,f\right]dt.$$

Thus we have

$$(L_nf)^{(p+2)}(x)=\frac{(n+1)^{n-p}\,n!\,(p+1)!}{n^n\,(n-p-1)!}(n-p-1)\cdot$$

$$\cdot\sum_{k=0}^{n-p-2}C_{k,p+2}(x)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}}\left[t,\,t+\frac{1}{n+1},\,\ldots,\,t+\frac{p+1}{n+1};\,f\left(\cdot+\frac{1}{n+1}\right)-f\right]dt$$

i. e.,

$$(L_nf)^{(p+2)}(x)=\frac{(n+1)^{n-p-1}\,n!\,(p+2)!}{n^n\,(n-p-2)!}\cdot$$

(4)
$$\cdot\sum_{k=0}^{n-p-2}C_{k,p+2}(x)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}}\left[t,\,t+\frac{1}{n+1},\,\ldots,\,t+\frac{p+2}{n+1};\,f\right]dt,$$

because

$$\left[t,\,t+\frac{1}{n+1},\,\ldots,\,t+\frac{p+1}{n+1};\,f\left(\cdot+\frac{1}{n+1}\right)-f\right]=$$

$$=\left[t,\,t+\frac{1}{n+1},\,\ldots,\,t+\frac{p+2}{n+1};\,f\right]-\left[t,\,t+\frac{1}{n+1},\,\ldots,\,t+\frac{p+1}{n+1};\,f\right]=$$

$$=\frac{n+1}{p+2}\left[t,\,t+\frac{1}{n+1},\,\ldots,\,t+\frac{p+2}{n+1};\,f\right].$$

In this manner the formula (3) is valid. In other words with substitution $t \to \frac{t+k}{n+1}$ we have

$$(L_n f)^{(p+2)}(x) = \frac{(n+1)^{n-p-2}\, n!\, (p+2)!}{n^n (n-p-2)!} \cdot$$

$$\cdot \sum_{k=0}^{n-p-2} C_{k,p+2}(x) \int_0^1 \left[\frac{t+k}{n+1}, \frac{t+k+1}{n+1}, \ldots, \frac{t+k+p+2}{n+1}; f\right] dt.$$

Now, because

$$(L_n f)^{(j)}\left(\frac{1}{2n+2}\right) = \frac{n!\, j!}{n^j (n-j)!} \int_0^1 \left[\frac{t}{n+1}, \frac{t+1}{n+1}, \ldots, \frac{t+j}{n+1}; f\right] dt$$

on account of Taylor expansion (1) follows. Further, $x \in K_1$ and $n \geqq \left[\frac{1}{2\varepsilon}\right]$ imply

$$x \in \left[\frac{1}{2n+2}, 1 - \frac{1}{2n+2}\right]$$

and

$$P_{r,k} \geqq 0 \quad \text{on} \quad K_1,$$

this is L_n, $n \in \mathbf{N}$ are positive operators. □

2. The convergence properties.

We prove the following

Lemma. *If* $e_j(y) := y^j$, $j = 0, 1, \ldots,$ *then we have*

$$L_n e_0 = e_0,$$

$$L_n e_1 = e_1,$$

$$L_n e_2 = \frac{n-1}{n} e_2 + \frac{1}{n} e_1 - \frac{5n+3}{12n(n+1)^2} e_0,$$

$$L_n e_3 = \frac{(n-1)(n-2)}{n^2} e_3 + \frac{3(n-1)}{n^2} e_2 - \frac{n^2-9n-6}{4n^2(n+1)^2} e_1 - \frac{2n+1}{4n^2(n+1)^2} e_0,$$

(5)

$$L_n e_4 = \frac{(n-1)(n-2)(n-3)}{n^3} e_4 + \frac{6(n-1)(n-2)}{n^3} e_3 + \frac{3(n-1)(n+3)(3n+2)}{2n^3(n+1)^2} e_2 -$$

$$- \frac{7n^2-7n-6}{2n^3(n+1)^2} e_1 + \frac{n^3-90n^2-105n-30}{80n^3(n+1)^4} e_0.$$

Proof. If we put

$$E_p^m := \sum_{k=0}^{m} k^p \binom{m}{k} a^k t^{n-k}, \qquad m = 0, 1, \ldots,$$

then we may write

$$E_o^m = (a+b)^m,$$

$$E_p^m = ma \sum_{k=1}^{m} k^{p-1} \frac{k}{m} \binom{m}{k} a^{k-1} b^{n-k}$$

$$= ma \sum_{k=0}^{m-1} (k+1)^{p-1} \binom{m-1}{k} a^k b^{m-1-k}$$

$$= ma \sum_{\nu=0}^{p-1} \binom{p-1}{\nu} E_\nu^{m-1},$$

$$p = 1, 2, \ldots .$$

Setting

$$m = n, \quad a = x - \frac{1}{2n+2}, \quad b = \frac{2n+1}{2n+2} - x$$

in above formulas we may write

$$L_n e_j = \frac{(n+1)^{n-j}}{n^n (j+1)} \sum_{\nu=0}^{j} \binom{j+1}{\nu} E_\nu^n .$$

By using this formula we can calculate (5). □

Theorem 2. *If* $L_n \; C(K) \to C(K_1)$, $n \in \mathbf{N}_\varepsilon$ *are defined as in* (1)–(2), *then for every* $f \in C(K)$ *we have*

$$\lim_{n \to +\infty} \|f - L_n f\|_1 = 0.$$

Proof. Let $H: = \langle e_0, e_1, e_2 \rangle$ be a set of polynomials of degree two. The well-known result of T. Popoviciu [11] asserts that it is sufficient to prove the convergence at any element from H in order to prove the pointwise convergence on the whole space $C(K_1)$ of a sequence of linear positive operators to the identity operator. This theorem was extended later by P. P. Korovkin [4] to the case when $H = \langle f_0, f_1, f_2 \rangle$ where $\{f_0, f_1, f_2\}$ is a Chebyshev system. Thus our theorem is proved if we show that

$$\lim_{n \to +\infty} \| e_j - L_n e_j \|_1 = 0, \qquad j = 0, 1, 2.$$

But this is immediate having in view the above lemma. □

Another proof may be performed as follows (see [6, 7]) where the convexity role in the approximation by linear positive operators is put in evidence.

Let $f \in C^{(2)}(K)$, $\quad m_f \leqq f'' \leqq M_f$

and

$$h_1(x): = f(x) - m_f \frac{x^2}{2}, \qquad h_2(x): = M_f \frac{x^2}{2} - f(x).$$

Taking into account that h_j, $j=1, 2$ are convex (non-concave) on K, the following inequalities are valid [7]

$$h_j(x) \leqq (L_n h_j)(x), \quad x \in K_1, \quad n \in \mathbf{N}_\varepsilon, \quad j=1, 2,$$

i. e.,

$$\frac{1}{2} m_f (L_n e_2 - e_2) \leqq L_n f - f \leqq \frac{1}{2} M_f (L_n e_2 - e_2).$$

Making use of (5) we get

$$\frac{1}{12(n+1)^2} \leqq L_n e_2 - e_2 = \frac{x(1-x)}{n} - \frac{5n+3}{12n(n+1)^2} \leqq \frac{3n+1}{12(n+1)^2}.$$

In such a way for $f \in C^{(2)}(K)$ and $n \in \mathbf{N}_\varepsilon$ we have

$$\frac{1}{24(n+1)^2} m_f \leqq L_n f - f \leqq \frac{1}{8(n+1)} M_f \quad \text{on} \quad K_1.$$

In other words the sequence of operators is pointwise convergent to the identity on a dense subset. But $\|L_n\| = 1$, $n \in \mathbf{N}_\varepsilon$, finishes the proof. □

Theorem 3. *If $\omega(f; \delta)$ is the modulus of continuity, then we have*

$$\|f - L_n f\|_1 < \frac{19}{16} \omega\left(f; \frac{1}{\sqrt{n+1}}\right).$$

Proof. Let $\Omega_{2m}(t, x) := (t-x)^{2m}$. According to a result of A. Lupaş and M. W. Müller [8] we may write

$$\|f - L_n f\|_1 \leqq \inf_{m=1,2,\dots} \{1 + \delta^{-2m} \|L_n \Omega_{2m}\|_1\} \omega(f; \delta), \quad \delta > 0. \tag{6}$$

Let us put

$$Y_n(x) := \left(x - \frac{1}{2n+2}\right)\left(\frac{2n+1}{2n+2} - x\right), \qquad x \in K_1.$$

By using the above lemma we obtain

$$\begin{aligned}
(L_n \Omega_4)(x) &= \sum_{k=0}^{n} P_{n,k}(x) \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} (t-x)^4 \, dt \\
&= \sum_{\nu=0}^{4} \binom{4}{\nu} (-1)^\nu x^{4-\nu} (L_n e_\nu)(x) \\
&= \frac{3(n-2)}{n^3} x^4 - \frac{6(n-2)}{n^3} x^3 + \frac{3(2n^3 + n^2 - 9n - 6)}{2n^3(n+1)^2} x^2 - \\
&\quad - \frac{3(n^2 - 3n - 2)}{2n^3(n+1)^2} x + \frac{n^3 - 90n^2 - 105n - 30}{80n^3(n+1)^4},
\end{aligned}$$

which may be written in the form

$$(L_n \Omega_4)(x) = \frac{3(n-2)}{n^3} Y_n^2(x) + \frac{3}{2n(n+1)^2} Y_n(x) + \frac{1}{80(n+1)^4}. \tag{7}$$

Further we have

$$\| L_n \Omega_4 \|_1 \leq \frac{3(n-2)}{n^3} \max_x Y_n^2(x) + \frac{3}{2n(n+1)^2} \max_x Y_n(x) + \frac{1}{80(n+1)^4} =$$

$$= \frac{3(n-2)}{n^3} Y_n^2\left(\frac{1}{2}\right) + \frac{3}{2n(n+1)^2} Y_n\left(\frac{1}{2}\right) + \frac{1}{80(n+1)^4}$$

$$= \frac{15n^2+1}{80(n+1)^4} < \frac{15(n+1)^2}{80(n+1)^4} < \frac{3}{16(n+1)^2}, \quad n=1, 2, \ldots .$$

If in (6) we select $m=2$, $\delta = \frac{1}{\sqrt{n+1}}$, the proof is complete. □

Theorem 4. *If $f \in C^{(1)}(K)$ and $\omega(f'; \delta)$ is the modulus of continuity of f', then we have*

$$\|f - L_n f\|_1 \leq \frac{11}{16\sqrt{n+1}} \omega\left(f'; \frac{1}{\sqrt{n+1}}\right).$$

Proof. Setting

$$e_x(t) := t - x, \quad \Delta_x(t) := f(t) - f(x)$$

we observe that there is at least a point $\theta := \theta(t, x)$ such that

$$\min\{x, t\} < \theta < \max\{x, t\}$$

and

$$\Delta_x(t) = e_x(t) f'(x) + D_x(t)$$

where

$$D_x(t) := e_x(t)[f'(\theta) - f'(x)].$$

But $(L_n e_x)(x) = 0$ and for $n \in \mathbf{N}_\varepsilon$ we find

$$|f(x) - (L_n f)(x)| \leq |(L_n \Delta_x)(x)| = |(L_n D_x)(x)| \leq (L_n |D_x|)(x). \tag{8}$$

Further, if we put $\lambda_x = \lambda_x(t) := \frac{|\theta(t, x) - x|}{\delta}$, $\delta > 0$, we have

$$\omega(f'; \lambda\delta) \leq \omega(f'; (1+[\lambda])\delta) \leq (1+[\lambda])\, \omega(f'; \delta)$$

as well as

$$|D_x(t)| = |e_x(t)| \cdot |f'(\theta) - f'(x)| \leq |e_x(t)|\, \omega(f'; \lambda_x \delta).$$

Using these results we conclude

$$(L_n |D_x|)(x) \leq \omega(f'; \delta)\, [(L_n |e_x|)(x) + (L_n |e_x| [\lambda_x])(x). \tag{9}$$

Schwartz's inequality enables us to write

$$(L_n |e_x|)(x) \leq \sqrt{(L_n \Omega_2)(x)} \leq \sqrt{\| L_n \Omega_2 \|}.$$

Now, we put

$$\chi(t):=\begin{cases}1, & |\theta-x|\geqq\delta\\ 0, & |\theta-x|<\delta\end{cases},\quad \delta>0.$$

Taking into account that $[\beta]\leqq\beta$, $|\theta-x|<|t-x|$, we have

$$|e_x(t)|\cdot[\lambda_x(t)]\leqq|e_x(t)|(\lambda_x(t))^3\chi(t)\leqq$$
$$\leqq|e_x(t)|(\lambda_x(t))^3=\delta^{-3}|t-x|\cdot|\theta-x|^3\leqq$$
$$\leqq\delta^{-3}(t-x)^4.$$

In this way

$$|e_x(t)|\cdot[\lambda_x(t)]\leqq\delta^{-3}\Omega_4(t,x).$$

Since L_n is a monotone operator, we conclude that

$$(L_n|e_x|[\lambda_x])(x)\leqq\delta^{-3}(L_n\Omega_4)(x).$$

From these inequalities and (9) becomes

$$(10)\qquad (L_n|D_x|)(x)\leqq\omega(f',\delta)\{\|L_n\Omega_2\|^{1/2}+\delta^{-3}\|L_n\Omega_4\|\}$$
$$\forall x\in K_1,\quad n\in\mathbf{N}_\varepsilon,\quad \delta>0.$$

If we select $\delta:=1/\sqrt{n+1}$, according to (5), (7), (8), (10), we have

$$|f(x)-(L_nf)(x)|\leqq\frac{11}{16\sqrt{n+1}}\,\omega\left(f';\frac{1}{\sqrt{n+1}}\right)$$

which proves the assertion. □

It is clear that in the proof we have not used the forms (1) or (2) of the operators, only the fact that they are linear, positive and that they preserve that the linear functions were applied. This fact was put in evidence in [8].

3. The representation of the remainder.

For x fixed on K_1 let $R_x:C(K)\to\mathbf{R}$ be the remainder defined as

$$R_{n,x}[f]:=(L_nf)(x)-f(x).$$

The equalities (5) furnish

$$R_{n,x}[e_0]=R_{n,x}[e_1]=0,$$
$$R_{n,x}[e_2]=\frac{x(1-x)}{n}-\frac{5n+3}{12n(n+1)^2}.$$

In [7] it is shown that if L is a linear positive operator defined on $C(K)$ with the properties

1° $Le_0=e_0$, $Le_1=e_1$,

2° $f\in C(K)$, $f\geqq 0$ on K, $f\neq 0$ implies $Lf>0$ on $K_2\subseteq K$,

then for any continuous (strictly) convex function $f:K\to\mathbf{R}$ holds

$$f<Lf \quad\text{on}\quad K_2.$$

We return to the operators L_n; therefore

$$R_{n,x}[f]>0 \quad \text{for} \quad n\in \mathbf{N}_\varepsilon, \quad x\in K_1:=[\varepsilon, 1-\varepsilon].$$

T. Popoviciu [12] has established that in these hypothesis the functional $R_{n,x}:C(K)\to \mathbf{R}$ has "a simple form", i. e., there is a constant Q and the distinct points $\alpha_{nx}, \beta_{nx}, \gamma_{nx}$ from K so that

$$R_{n,x}[f]=Q\cdot[\alpha_{nx}, \beta_{nx}, \gamma_{nx}; f],$$

Q being independent of f. For $f:=e_2$ we find

$$Q=R_{n,x}[e_2].$$

In this manner we have proved

Theorem 5. *Let $f\in C(K)$ and L_n, $n\in \mathbf{N}_\varepsilon$ be defined by* (1). *For each $x\in K_1$ there exist the distinct points $\alpha_{nx}, \beta_{nx}, \gamma_{nx}$ such that*

$$(L_n f)(x)-f(x)=\left(\frac{x(1-x)}{n}-\frac{5n+3}{12n(n+1)^2}\right)\cdot[\alpha_{nx}, \beta_{nx}, \gamma_{nx}; f].$$

Another kind of representation theorem is the following asymptotic formula

Theorem 6. *Let us suppose that $f\in C(K)$, f'' exists at the point $x_0\in K_1$. Then*

$$\lim_{n\to+\infty} n\{(L_n f)(x_0)-f(x_0)\}=\frac{x_0(1-x_0)}{2}f''(x_0).$$

The proof is similar to the well-known result of E. Voronovskaja for the Bernstein operators [5]. Likewise it may be deduced from the general theorem obtained in [9] by R. G. Mamedov.

Other topics, as converse theorems, saturation theorems, may be investigated by means of the methods used by J. Karamata and M. Vuilleumier [3], De Vore [2]. For instance we note that the saturation class is

$$\text{Sat. kl. } [L_n]=\left\{f\in C(K) \mid \|f-L_n f\|_1=O\left(\frac{1}{n}\right)\right\}=\text{Lip}^{(1)}\,1\,(K)$$

where $\text{Lip}^{(1)}\,1\,(K)$ is the subspace of $C(K)$ formed with those functions which have a derivative $f'\in \text{Lip } 1$ on K.

4. The convexity preservation.

As we have seen the operators L_n, $n\in \mathbf{N}_\varepsilon$, have the interesting property that

$$f\leqq L_n f \text{ on } K_1$$

for $f\in C(K)$ non-concave on K. Another result is

Theorem 7. *The operators L_n, $n\in \mathbf{N}_\varepsilon$, preserve the shape of the function. More precisely, if $f\in C(K)$ is a non-concave function of the order $p+1$, then $L_n f$, $n\in \mathbf{N}_\varepsilon$, are nonconcave on K_1 of the same order, $p=-2, -1, 0, \ldots$.*

Proof. According to (4) the inequalities

$$\left[\frac{t+k}{n+1}, \frac{t+k+1}{n+1}, \ldots, \frac{t+k+p+2}{n+1}; f\right] \geqq 0$$

$$C_{k,p+2}(x) \geqq 0$$

$$k=0,1,\ldots, n-p-2; \quad x \in K_1, \quad t \in [0,1]$$

certifies the validity of inequality

$$\frac{d^{p+2}}{dx^{p+2}}(L_n f)(x) \geqq 0 \quad \forall x \in K_1,$$

and the theorem is proved. □

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