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A NOTE ON A CLASS OF NOETHERIAN RINGS

Ravinder Kumar and Kanchan Manaktala

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Introduction: All rings considered here are commutative with identity 1 ($\neq 0$). Further, to avoid trivial considerations we assume that R is distinct from its total quotient ring. For each ideal A of R let A^* denote the subring of R generated by $A \cup \{1\}$. Let $A(R) = \{A^* | A$ is a proper ideal of $R\}$. The motivation for the present note stems from our starting Lemma and the fact that for a group G if $[S]$ denotes the subgroup of G generated by $S \subseteq G$ then card $\{[S]/\{e\} < S < G\} = 1$ if and only if $G \cong \frac{Z}{(p)}$, where Z is the additive group of integers and p, a prime number. Further, we notice that card $A(R) = 1$ implies that R is Noetherian. In this note we classify all those rings R for which card $A(R) = 1$. A point of significance to be noticed is that rings R with card $A(R) = 1$ can be conveniently expressed in terms of the ring of integers and rings associated to it.

Preliminaries: A ring R is said to be a multiplication ring if for any two ideals A and B of R with $A \subset B$ there is another ideal C of R such that $A = BC$. In other words we want ideal theoretic division to coincide with number theoretic division. An ideal A of R is said to be proper if $(0) < A < R$ where $<$ denotes proper set theoretic containment. An ideal A is said to be simple if there is no ideal properly between A and A^2. An integral domain D is said to be a (KE)-domain if for each proper ideal A of D $A^* = D$ and D is a Dedekind domain [4]. A ring R is said to be an (ME)-ring if the class $A(R)$ is a subclass of the subrings of R which are multiplication rings [5]. For multiplication rings we refer to [2] and for (KE)-domains and (ME)-rings we refer to [4, 5]. A ring R is said to be special primary if it has a unique nonzero maximal ideal which is i) nilpotent and ii) simple. Z- denotes the ring of integers. Other notations and terminologies are taken from [3].

1. Integral Domains with Card $A(R) = 1$

Lemma 1.1: *Let D be an integral domain. D is a (KE)-domain if and only if card $A(D) = 1$.*

Proof. If D is a (KE)-domain then clearly card $A(D) = 1$. To prove the converse we first show that the condition that (KE)-domain D should be a Dedekind domain is redundant. Thus, let $A^* = D$ for each proper ideal A of D.

Now for each proper ideal A of D, $\frac{D}{A}=\frac{A^*}{A}\cong\frac{Z}{(m)}$ where m is an integer. This implies that proper homomorphic images of D are Noetherian and so is D. To show that D is a Dedekind domain it is now sufficient to prove that each proper prime ideal of D is simple [1, Theorem 6].

Let P be a proper prime ideal of D. Clearly, $(0)<P^2<P$. Further, $\frac{D}{P}\cong Z$ or $\frac{Z}{(p)}$, p prime. Also $\frac{D}{p^2}\cong\frac{Z}{(n)}$, $n\neq 0$. But this contradicts $\frac{D}{p}\cong Z$. Again $\frac{D}{P}\cong\frac{Z}{(p)}$ implies that $\frac{D}{P^2}\cong\frac{Z}{(p^2)}$. This yields that P is a simple prime ideal of D. Thus, D is Dedekind and therefore a (KE)-domain.

Next assume that card $A(D)=1$. Let $A(D)=\{E\}$. Now let A be a proper ideal of D. We show that A^* is a (KE)-domain. Let B be a proper ideal of A^*. Then $E=(BA)^*\subset B^*\subset A^*=E$. This shows that A^* is a (KE)-domain for each proper ideal A of D. Invoking Theorem 2 in [5], we get that D is a (KE)-domain. This proves the Lemma.

Combining this Lemma with Theorem 14 [4] we get the following Proposition which shows that domain with card $A(D)=1$ can be nicely related to the ring of integers.

Proposition 1.2: Let D be an integral domain. Card $A(D)=1$ if and only if there is a multiplicative set S of integers, $0\notin S$ such that

α) Z_S is imbeddable in D,

β) there is a $1-1$ correspondence between the proper ideals A of D and those of Z_S given by $A\leftrightarrow A\cap Z_S$,

γ) for each proper prime ideal P of D

$$\frac{D}{P}\cong\frac{Z_S}{P\cap Z_S}.$$

2. Rings with Zero Divisors and Card $A(R)=1$

In this section R denotes a ring which has nonzero divisors of zero. Then we have the following properties.

Property 1: Let card $A(R)<\aleph_0$. Then R is Noetherian (here $\aleph_0$ denotes the cardinality of the set of natural numbers).

Property 2: If Card $A(R)=1$ then each proper homomorphic image of R is a multiplication ring.

Now we record a useful Lemma due to Gilmer and Mott [2].

Lemma 2.1 *Let A be a simple ideal of a ring R. Then the only ideals of R contained between any two powers of A are themselves the intermediate powers of A.*

First we consider special primary rings with card $A(R)=1$.

Lemma 2.2: *Let R be a special primary ring such that card* $A(R)=1$. *Then R is one of the following:*

i) $R \cong \frac{Z}{(p^n)}$, *where p is a prime number and n an integer* $\geqslant 3$.

ii) *R is a special primary ring with degree of nilpotency of its maximal ideal equal to* 2.

Proof. Let K be the degree of nilpotency of M, the unique maximal ideal of R. Suppose $K \geqslant 3$. Let $A(R)=\{S\}$. M is a simple maximal ideal of S too. Thus for each proper ideal A, A^* is a special primary ring. Hence R is a quasi-local (ME)-ring. Invoking Theorem 4 [5] we find that there is a prime number p such that $\frac{R}{M} \cong \frac{Z}{(P)}$, card $R=p^n$, $n \geqslant 3$, and if $n \geqslant 4$ then R is a special primary ring isomorphic to $\frac{Z}{(p^n)}$. Thus it is enough to show that $R \cong \frac{Z}{(p^3)}$ whenever $n=3$ and $M^2 \neq (0)$. This will be proved if characteristic of R equals p^3.

Case 1: Suppose characteristic of R is p. Then $\frac{S}{M} \cong \frac{Z}{(p)}$. Let $(X)=M$ in $S(X^2)^*=(X)^*$. Thus $X=\alpha X^2+\beta$, with α, β invertible. This gives X is invertible which it is not.

Case 2: Suppose characteristic of R is p^2. Here, again, we get that $X=\alpha X^2+\beta$, where α is invertible and β is of the form $m\ p.\ 1$, $1 \leqslant m \leqslant p-1$. This gives that

$X^2=(\alpha X^2+\beta)^2=2\alpha mpX^2$. This yields that $X^2=0$. But then k should be 2, which is again a contradiction.

Thus the proof of the Lemma is complete.

The structure of Noetherian multiplication rings has been obtained by many authors as direct sums of Dedekind domains and special primary rings. For one such discussion we refer to Gilmer and Mott [2] By Property 2 our rings have their proper homomorphic images as multiplication rings. The structure of such rings has been obtained by Wood [6]. To simplify the proof of our final Lemma we shall need the structure of Noetherian (ME)-rings which has been obtained by Singh and Kumar [5].

Lemma 2.3. *Let R be a ring which is not an integral domain. If card* $A(R)=1$ *then R is one of the following;*

i) *R is a special primary with the degree of nilpotency of its maximal ideal equal to* 2,

ii) $R \cong \frac{Z}{(p)} \oplus \frac{Z}{(p)}$ *where p is a prime,*

iii) *R is a homomorphic image of the ring of integers.*

Proof. Case 1: *R is a primary ring.* Let M be the unique nonzero prime ideal of R. Let $A(R)=\{S\}$. In view of the preceding remarks we get by Wood [6] that either R is a special primary ring in which case it is done by Lemma 2.2 or R has the following properties:

α) M is not simple, β) $M^2=(0)$,

γ) M is generated by two of its noncomparable elements X and Y which exist.

Suppose R is a ring with properties α), β) and γ). We have $\frac{S}{M}\cong\frac{Z}{(p)}$ for some prime p. Clearly, $(X)^*=(Y)^*$ yields that $X=rY+nl$ where r is in S and n is an integer of the form pq. $q=p$ yields that $(X)\subset(Y)$ which is not true. Thus $1\leqslant q<p$. But this gives an element z in M such that $\frac{S}{(z)}\cong\frac{Z}{(p)}$ so that $M=(z)$ which is again a contradiction. This settles our claim in this case.

Case 2: *R is not a primary ring.* Invoking the aforesaid result of Wood we find that R is itself a multiplication ring. R being Noetherian by Property 1, R is a direct sum of Dedekind domains and special primary rings. It is enough to assume that R has at least two direct summands in its representation as direct sum. Clearly, $R=S$. Thus R is a Noetherian (ME)-ring. By Singh and Kumar [5], R is one of the following

λ) $R=S_0\oplus T$, where S_0 is a von Neumann regular ring of finite characteristic and T is a (KE)-domain,

μ) R is a von Neumann regular ring,

ν) $R=S_0\oplus R_1\oplus\cdots\oplus R_t$ where S_0 is a von Neumann regular ring of finite characteristic and $R_1, R_2, \ldots, R_t$ are local (ME)-rings none of which is a domain and $0(R_i)$'s are powers of distinct primes.

We take these cases one by one.

λ) Here $R\cong\frac{Z}{(m)}\oplus Z$. It is clear that card $A(R)>1$ in this case.

μ) There exists an idempotent e distinct from 0 and 1 such that $R=$ $=eR\oplus(1-e)R$. In this case $R\cong\frac{Z}{(m_1)}\oplus\frac{Z}{(m_2)}$. But R being Noetherian von Neumann regular ring m_1 and m_2 are primes, so that R is one of the desired rings.

ν) In this case $R\cong\frac{Z}{(m)}\oplus\frac{Z}{(p_1)^{\lambda_1}}\oplus\cdots\oplus\frac{Z}{(p_t)^{\lambda_t}}$ where p_i are distinct primes and m is product of distinct primes. Further, it can be seen that none of the p_i's divides m. This R is a homomorphic image of the ring of integers. This proves the Lemma.

Now we are ready to state the major result of this note the truth of which can be easily established by patching up Proposition 1.2 Lemma 2.2 and Lemma 2.3.

Theorem: *Let R be a commutative ring with unity such that R is distinct from its total quotient ring. The card $A(R)=1$ if and only if R is one of the following:*

i) *R is a special primary ring with the degree of nilpotency of its maximal ideal equal to 2.*

ii) *R is a homomorphic image of the ring of integers,*

iii) $R=\frac{Z}{(p)}\oplus\frac{Z}{(p)}$, *$p$ prime,*

iv) *R is an integral domain and a suitable multiplicative set S of nonzero integers can be found such that.*

α) *Z_S is imbeddable in R,* β) *there is a $1-1$ correspondence between the proper ideals A of R and those of Z_S given by $A \leftrightarrow A \cap Z_S$,*

γ) *for each proper prime ideal P of R. $R/P \cong Z_S/P \cap Z_S$.*

Remark: Except when R happens to be a ring of type i) $A^*=R$ for all proper ideals A of R. Examples can be easily constructed to show that A^* may not equal R in case i).

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Ramjas College, Delhi University, India
Gargi College, Delhi University, India