PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 19 (33), 1975, pp 81—82

ADDITIONS TO KAMKE'S TREATISE, VI: A NONLINEAR SECOND ORDER EQUATION

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(Received June 6, 1975)

1. The following nonlinear second order differential equations, together with their solutions, are recorded in Kamke's collection [1]

$$y''+f(x)\,y'+g(y)\,(y')^2=0, \tag{1}$$

$$y''+f(x)\,y'-\frac{F'(y)}{F(y)}(y')^2+h(x)\,F(y)=0, \tag{2}$$

as equations 6.51 and 6.52, respectively.

Equation (1) is often called Liouville's equation.

Setting $g(y)=-F'(y)/F(y)$, equation (2) takes the following form:

$$y''+f(x)\,y'+g(y)\,(y')^2+h(x)\,e^{-\int g(y)\,dy}=0. \tag{3}$$

The object of this note is to solve a nonlinear second order equation which contains equations (1) and (3) as particular cases.

2. The equation in question is

$$y''+f(x)\,y'+g(y)\,(y')^2+h(x)\,e^{(k-1)\int g(y)\,dy}\,(y')^k=0 \tag{4}$$

where $k\in R$.

For $h(x)\equiv 0$ we get (1), and for $k=0$ we get (3).

We first consider the equation

$$y''+g(y)\,(y')^2=0,$$

which after integration yields

$$y'=Ce^{-\int g(y)dy}, \tag{5}$$

where C is an arbitrary constant.

Suppose that C is a differentiable function of x. Then, differentiating (5), we find

$$y''=C'(x)\,e^{-\int g(y)\,dy}-C(x)\,g(y)\,y'\,e^{-\int g(y)\,dy}. \tag{6}$$

Substitute (5) and (6) into (4) to obtain

(7) $$C'(x)+f(x)C(x)+h(x)C(x)^k=0.$$

Equation (7) is a Bernoulli type equation for $C(x)$.

The general solution of (4) is then found from (5):

$$\int e^{\int g(y)\,dy}dy=\int C(x)\,dx+B,$$

where $C(x)$ is the general solution of (7), containing one arbitrary constant, and B is the other arbitrary constant.

3. We give an other particular case of (4). Namely, for $g(y)=-1/y$ and $k=0$, equation (4) becomes

(8) $$y''+f(x)y'-y^{-1}(y')^2+h(x)y=0.$$

In his classical work [2] Painlevé (see also [3], p. 60) solved the equation

(9) $$y''+f(x)y'+ay^{-1}(y')^2+h(x)y=0,$$

provided that $a\neq -1$.

For $a=-1$, Painlevé's equation (9) becomes (8), and hence it can also be solved.

REFERENCES

[1] Э. Камке, *Справочник по обыкновенным дифференциальным уравнениям*, Москва 1971.

[2] P. Painlevé, *Sur les équations différentielles du second ordre et d'ordre supérieur dont l'intégrale générale est uniforme*, Acta Math. 25 (1902), 1-85.

[3] W. F. Ames, *Nonlinear ordinary differential equations in transport processes*, New York — London 1968.