PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 19 (33), 1975 pp. 77—80

TRANSFORMATIONS OF MEASURABLE SETS BY AUTOMORPHISM GROUPS

S. S. Jou

(Received March 27, 1974)

1. The purpose of this note is to extend the Theorem of [3] into more general settings and at the same time generalize the Theorems of [5] into more general representations.

We follow the notations of [3] and the definitions are standard by [1]. G always denotes locally compact group (regarded Hausdorff by [1]) and μ denotes a left Haar measure on G. R^+ denotes the set of all positive real numbers and C denotes the set of all complex numbers. Suppose $f: G \to C$. The left and right translations of f are denoted by ${}_af$ and f_a for any $a \in G$; namely ${}_af(x) = f(ax)$ and $f_a(x) = f(xa)$ for all $x \in G$. $L^1(G)$ denotes the Banach space of all complex-valued integrable functions and $\| \ \|$ is used to denote the L^1-norm. Suppose $T: G \to G$ and $f: G \to C$. Denote $f_T = f \cdot T: G \xrightarrow{T} G \xrightarrow{f} C$. We use l^A to denote the characteristic function of the subset A of G; namely $l^A(x) = l$ or 0 depending on $x \in A$ or otherwise. $l^A{}_T$ is understood to be $l^A \cdot T$.

Let $\mathfrak{S}(G)$ denote the collection of all automorphisms of G ([1], page 426). In the following, we will assume as open neighbourhoods of the identity l_G of $\mathfrak{S}(G)$, the collection $\{\mathcal{B}(F, U)\}$, where for a compact subset F of G and U an open neighbourhood of the identity e of G, $\mathcal{B}(F, v)$ is the set of all $\sigma \in \mathfrak{S}(G)$ such that $\sigma(a) \in Ua$ and $\sigma^{-1}(a) \in Ua$ for all $a \in F$. Then $\mathfrak{S}(G)$ is a topological group with the topology of open sets of the form $\cup \{\sigma \mathcal{B}(F, U)\}$ where $\sigma \in \mathfrak{S}(G)$ ([1], page 427). It is easy to see that $(x, \sigma) \to \sigma(x)$ is a contionuous mapping of $G \times \mathfrak{S}(G)$ into G. Hence $\mathfrak{S}(G)$ is a transformation group on G.

Theorem: *Let $A, B_1, B_2, \ldots, B_n$ be subsets of a locally compact group G of strictly positive finite Haar measure. Then the mapping*

$$(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n, T_1, T_2, \ldots, T_n) \to$$
$$\to \mu [A \cap L_{x_1} R_{y_1} T_1 (B_1) \cap L_{x_2} R_{y_2} T_2 (B_2) \cap \cdots \cap L_{x_n} R_{y_n} T_n B_n)]$$

is a nontrivial continuous mapping of $G^{2n} \times [\mathfrak{S}(G)]^n$ into R^+ where $x_i \in G$, $y_i \in G$, $T_i \in \mathfrak{S}(G)$, $L_{x_i}: G \to G$ and $R_{y_i}: G \to G$ such that $L_{x_i}(z) = x_i z$ and $R_{y_i}(z) = zy_i$ for all $z \in G$ and $| \leqslant i \leqslant n$. The Theorem is irrelevant of the order of L_i, R_{y_i} and T_i.

2. The following three lemmas are important to the proof of the Theorem.

Lemma 1. *There exists a continuous homomorphism* ρ *of* $\mathfrak{S}(G)$ *into the multiplicative group* $R^+/\{0\}$ *such that for any* $f \in L^1(G)$, *then*

$$\int_G f_T(x)\, d\mu(x) = \rho(T^{-1}) \int_G f(x)\, d\mu(x)$$

for any $T \in \mathfrak{S}(G)$. *Hence* $f_T \in L^1(G)$ *if* $f \in L^1(G)$ *and* $T \in \mathfrak{S}(G)$.

Proof: As for the existence and continuity of ρ see [2]. The second part of this Lemma is easy.

Lemma 2: *Suppose* $f \in L^1(G)$ *and* $\{T_\gamma\}$ *is a net in* $\mathfrak{S}(G)$ *such that* $T_\gamma \to S \in \mathfrak{S}(G)$. *Then*

$$\lim_\gamma \| f_{T_\gamma} - f_S \| = 0.$$

Proof Given $\varepsilon > 0$, choose a continuous function g with compact support K such that

$$\| f - g \| < \frac{\varepsilon}{4\rho(S^{-1})}.$$

Then

$$\| f_{T_\gamma} - f_S \| \leq \| f_{T_\gamma} - g_{T_\gamma} \| + \| g_{T_\gamma} - g_S \| + \| g_S - f_S \|$$

$$\leq \rho(T_\gamma^{-1}) \| f - g \| + \int_G | g(T_\gamma(x)) - g(S(x)) |\, d\mu(x) + \rho(S^{-1}) \| g - f \|$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \varepsilon, \text{ as } T_\gamma \text{ is sufficiently close to } S.$$

For

$$I = \int_G | g(T_\gamma(x) - g(S(x)) |\, d\mu(x)$$

$$= \rho(S^{-1}) \int_G | g(T_\gamma S^{-1}(x)) - g(x) |\, d\mu(x)$$

and we can find an open neighbourhood $\overline{W}$ of e with compact closure $\overline{W}$ such that $ST_\gamma^{-1}(K) \subset \overline{W}K$ whenever T_γ is sufficiently close to S. Hence we have

$$I = \rho(S^{-1}) \int_{\overline{W}K} | g(T_\gamma S^{-1}(x)) - g(x) |\, d\mu(x)$$

$$\leq \rho(S^{-1})\, \varepsilon_1\, \mu(\overline{W}K) \leq \frac{\varepsilon}{4}, \text{ (by letting } \varepsilon_1 < \frac{\varepsilon}{4\rho(S^{-1})\,\mu(\overline{W}K)},$$

because for any $x \in \overline{W}K$, $[T_\gamma S^{-1}(x)\, x^{-1}]$ is sufficiently close to e whenever T_γ is sufficiently close to S.

Lemma 3: *Let $f\in L^1(G)$. Then $(x, y, T)\to {}_x(f_T)_y$ is a continuous mapping of $G^2\times\mathfrak{S}(G)$ into $L^1(G)$.*

Proof. For any $(u, v, S)\in G^2\times\mathfrak{S}(G)$ we have

$$\|{}_x(f_T)_y-{}_u(f_S)_v\|\leq\|{}_x(f_T)_y-{}_x(f_S)_y\|+\|{}_x(f_S)_y-{}_u(f_S)_v\|.$$

By Lemma 1 and Lemma 1 of [3],

$$\|{}_x(f_S)_y-{}_u(f_S)_v\|\to 0 \quad\text{as}\quad (x, y)\to(u, v)$$

and

$$\|{}_x(f_T)_y-{}_x(f_S)_y\|=\Delta(y^{-1})\,\|f_T-f_S\|\to 0$$

as $(x, y, T)\to(u, v, S)$ by Lemma 2 and the continuity of the modular function Δ of G (see [1], [3]).

Hence $\|{}_x(f_T)_y-{}_u(f_S)_v\|\to 0$ as $(x, y, T)\to(u, v, S)$.

Proof of the Theorem: For any fixed $(u_1, u_2, \dots, u_n, v_1, \dots, v_n, S_1, S_2, \dots, S_n)$ in $G^{2n}\times[\mathfrak{S}(G)]^n$, and any $(x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_n, T_1, T_2, \dots, T_n)$ in

$$G^{2n}\times[\mathfrak{S}(G)]^n,\ |\mu(A\cap x_1T_1(B_1)y_1\cap\cdots\cap x_nT_n(B_n)y_n)-\mu(A\cap u_1S_1(B_1)v_1$$

$$\cap\cdots\cdots\cap u_nS_n(B_n)v_n)\leq\sum_{j=1}^{n}\|l^{x_jT_j(B_j)y_j}-l^{u_jS_j(B_j)v_j}\|$$

(by Lemma 2 [3])

$$\leq\sum_{j=1}^{n}\Big\|{}_{x_j^{-1}}\Big(l_{T_j^{-1}}^{B_j}\Big)_{y_j^{-1}}-{}_{u_j^{-1}}\Big(l_{S_j^{-1}}^{B_j}\Big)_{v_j^{-1}}\Big\|\to 0$$

as $(x_j, y_j, T_j)\to(u_j, v_j, S_j)$ for $j=1, 2, \dots, n$, by lemma 3. Hence the mapping is continuous. By Lemma 3 of [3], it is easy to see that the mapping is non-trivial.

If we change the order of the composition of L_{x_i}, R_{y_i} and T_i, then the mapping is also continuous and not identically zero since $\sigma L_x=L_{\sigma(x)}\sigma$, $\sigma R_y=R_{\sigma(y)}\sigma$ and $(x, \sigma)\to\sigma(x)$ is jointly continuous for any $\sigma\in\mathfrak{S}(G)$ and $x, y\in G$. Now the proof is complete.

Corollary: *Suppose $\infty>\mu(A)>0$, $\infty>\mu(B)>0$. Then for any $\sigma_0\in\mathfrak{S}(G)$ there exists an open neighbourhood of W of σ_0 such that $\bigcap_{\sigma\in W}A[\sigma(B^{-1})]$ contains an open subset of G.*

By the above corollary, if we take $\sigma_0=I_G$ then AB^{-1} will contain an open subset of G. This is practically the Steinhaus Theorem [7] in the case of locally compact group.

Under composition, we see that the general linear group $Gl(n, R)$ of non-singular linear transformations of R^n with topology inherited from the uniform topology on the collection $\mathcal{L}(R^n, R^n)$ of all linear transformations is a topological group and $Gl(n, R)$ is topological isomorphic to $\mathfrak{S}(R^n)$. Hence it is easy to get all the consequences of [5] from our Theorem.

REFERENCES

[1] E. Hewitt and K. A. Ross, *Abstract harmonic analysis*, Springer-Verlag, 1963 (vol. 1).

[2] J. Braconnier, *Sur les groupes topologiques localement compacts*, J. Math. Pures Appl., N. S. 27, 1—85 (1948), (see page 75—76).

[3] S. S. Jou, *Transformation of measurable sets*, Glasnik Mat. **8** (28), 1973, 81—84.

[4] L. H. Loomis, *An introduction to harmonic analysis*, Princeton, N. J., D. Van Nostrand Co., 1953.

[5] M. Pal, *On Certain transformations of sets of positive measure*, Publication de l'institut Mathématique, Nouvelle série, **14** (28), 1972, 123—128.

[6] D, Montgomery and L. Zippin, *Topological transformation groups*, New York, Interscience Publications, Inc. 1955.

[7] H. Steinhaus, *Sur les distances des points des ensembles, de mesure positive* Fund. Math., **1** (1920), 93—104.

Universidad de los Andes
Mérida, Venezuela