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THE FUNCTIONALS OF THE KIND OF BANACH LIMITS

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Introduction

S. Banach ([1], p. 34) defined generalized limits (now commonly referred to as Banach limits) as positive, linear, shift-invariant functionals on the space of all bounded real-valued functions $t \to f(t)$, $t \geqslant 0$, which assign the value 1 to the constant function $t \to f(t) = 1$, $t \geqslant 0$.

It is the object of this paper to introduce a slightly more general family of functionals (than the Banach limits) defined on the vector space of all bounded functions $t \to f(t)$, $t \geqslant 0$, with values at an arbitrary real normed space.

In the sequel the following notation is used:

(i) X denotes an arbitrary real normed space,

(ii) E denotes the real vector space of all bounded functions $t \to f(t)$, $t \geqslant 0$, with values at X,

(iii) m denotes the real vector space of all bounded sequences of points of X.

Existential theorems

Let us put

$$p(f) = \inf_{a_i} \left\{ \overline{\lim_{t \to \infty}} \frac{1}{n} \left\| \sum_{i=1}^{n} f(t + a_i) \right\| \right\}, \qquad f \in E, \tag{1}$$

where the infimum is taken over all possible choices of natural numbers n and all nonnegative real numbers $a_1, a_2, \ldots, a_n$.

The functional $f \to p(f)$, $f \in E$, is seen to be real-valued.

L e m m a 1. *Let f and g be two arbitrary functions from E and let us suppose that* $\lim_{t \to \infty} g(t) = 0 \in X$. *Then*

$$p(f+g) = p(f). \tag{2}$$

P r o o f. For any $\varepsilon > 0$ there exists a number T so that $\|g(t)\| < \varepsilon$, $t \geqslant T$. Hence, for all $n = 1, 2, \ldots$, all $a_i \geqslant 0$ and all $t \geqslant T$

$$\frac{1}{n} \left\| \sum_{i=1}^{n} [f(t + a_i) + g(t + a_i)] \right\| \leqslant \frac{1}{n} \left\| \sum_{i=1}^{n} f(t + a_i) \right\| + \varepsilon,$$

which implies

$$\inf_{a_i}\left\{\overline{\lim_{t\to\infty}}\frac{1}{n}\left\|\sum_{i=1}^{n}[f(t+a_i)+g(t+a_i)]\right\|\right\}\leqslant\inf_{a_i}\left\{\overline{\lim_{t\to\infty}}\frac{1}{n}\left\|\sum_{i=1}^{n}f(t+a_i)\right\|\right\}+\varepsilon.$$

Since $\varepsilon>0$ is arbitrary, by (1), we have $p(f+g)\leqslant p(f)$. Hence also we have $p(f)=p((f+g)+(-g))\leqslant p(f+g)$. Therefore, $p(f+g)=p(f)$.

Lemma 2. *If for a function* $h\in E$, $\lim_{t\to\infty}h(t)=s\in X$, *then*

$$p(h)=\|s\|. \tag{3}$$

Proof. Let us put $h(t)=f(t)+g(t)$, $t\geqslant 0$, with $f(t)=s\in X$, $t\geqslant 0$, and $\lim_{t\to\infty}g(t)=0\in X$. Then, by (1) and (2), $p(h)=p(f+g)=\|s\|$, which completes the proof.

We can now proceed to the following statement.

Theorem 1. *Let E be the real vector space of all bounded functions $t\to f(t)$, $t\geqslant 0$, with values at a real normed space X and let the functional $f\to p(f)$, $f\in E$, be defined by means of* (1). *Then there exists a (nonunique) functional L defined on the whole space E with the following properties*:

(i) $L(af+bg)=aL(f)+bL(g)\ (a, b\in R,\ f, g\in E)$;

(ii) $f_a(t)=f(t+a)$, $t\geqslant 0$, *implies* $L(f_a)=L(f)\ (a>0)$;

(iii) $|L(f)|\leqslant p(f)\ (f\in E)$.

Also, let $\mathscr{L}$ denote the family of all functional L defined on E satisfying the conditions (i)—(iii). *Then for all $f\in E$*

(iv) $(\forall L\in\mathscr{L})\,L(f)=0$ *if and only if* $p(f)=0$

Sketch of the proof. It is evident that $p(af)=|a|\cdot p(f)$, $a\in R$, $f\in E$. Also, it is not difficult (see [2], p. 191—192) to prove that $p(f+g)\leqslant p(f)+p(g)$, $f, g\in E$. Hence, $f\to p(f)$, $f\in E$, is the symmetric convex functional on E. According to the consequence of the Hahn-Banach theorem from [1] (see also the Exercise 11.2 from [2], p. 187), there exists a (nonunique) linear functional L defined on the whole space E so that $|L(f)\leqslant p(f)$, $f\in E$, that is the functional L satisfying the conditions (i) and (iii) of the theorem Now, for any $r>0$ and $f\in E$ let us put

$$g_r=f_r-f,\quad s_i=(i-1)\,r,\quad i=1, 2, \ldots.$$

Then for all $n=1, 2, \ldots$ ‘ and all $t\geqslant 0$ we have

$$\frac{1}{n}\left\|\sum_{i=1}^{n}g_r(t+s_i)\right\|\leqslant\frac{1}{n}\|f(t+nr)-f(t)\|\leqslant\frac{2}{n}\sup_t\|f(t)\|.$$

Hence, for arbitrary $\varepsilon>0$ there exists a natural number N so that

$$\frac{1}{N}\left\|\sum_{i=1}^{N}g_r(t+s_i)\right\|<\varepsilon,\quad t\geqslant 0,$$

which implies

$$\overline{\lim_{t\to\infty}} \frac{1}{N} \left\| \sum_{i=1}^{N} g_r(t+s_i) \right\| \leqslant \varepsilon,$$

and

$$\inf_{a_i} \left\{ \overline{\lim_{t\to\infty}} \frac{1}{n} \left\| \sum_{i=1}^{n} g_r(t+a_i) \right\| \right\} \leqslant \varepsilon .$$

Since $\varepsilon > 0$ is arbitrary, we have

$$p(g_r) = \inf_{a_i} \left\{ \overline{\lim_{t\to\infty}} \frac{1}{n} \left\| \sum_{i=1}^{n} g_r(t+a_i) \right\| \right\} = 0.$$

Hence, by the conditions (i) and (iii) of the theorem, we have

$$0 = L(g_r) = L(f_r - f) = L(f_r) - L(f), \text{ or } L(f_r) = L(f),$$

that is that the condition (ii) of the theorem is satisfied, and we have proved the existence of a functional L satisfying the conditions (i)—(iii) of the theorem.

Now let E_0 be the subspace (of the space E) of all functions $f \in E$ having $\lim_{t\to\infty} f(t) = 0 \in X$. Then it is clear that

(4) $$p(f) = L(f) = 0, \quad f \in E_0 .$$

Let us put $h(t) = s \in X$, $t \geqslant 0$ $(s \neq 0)$. Then $h \in E \setminus E_0$ and $p(h) = \| s \| > 0$. Hence, the functional $f \to L(f)$, $f \in E_0$, can be extended to the subspace (of the space E) spanned by E_0 and h by defining the values at $f + ah$ $(f \in E_0$, $a \in R)$ to be $L(f) + ac$, where c is any element of the interval $[-p(h), p(h)]$. (This is a consequence of the proof of the Hahn-Banach theorem and (4)). Hence, there exist the functionals on E satisfying the conditions (i)—(iii) of the theorem with distinct values at h. Consequently, L is not a unique functional (satisfying the conditions (i)—(iii)).

Last, to complete the proof of theorem, it only remains to show that the condition (iv) of the theorem is satisfied. Indeed, let $\mathscr{L}$ denote the family of all functionals $f \to L(f)$, $f \in E$, satisfying the conditions (i)—(iii) of the theorem. Then, by the condition (iii), $p(f) = 0$, $f \in E$, implies $L(f) = 0, f \in E$ for every $L \in \mathscr{L}$. Conversely, $L(f) = 0$, $F \in E$, for every $L \in \mathscr{L}$ implies $p(f) = 0, f \in E$ because if $p(f) > 0$ for some $f \in E$, then, as in the proof above, there must exist at least two functionals satisfying the conditions (i)—(iii) with distinct values at f, which is in contradiction to $L(f) = 0$ for every $L \in \mathscr{L}$. The proof is now complete.

Let m be the real vector space of all bounded sequences of points from an arbitrary normed space X and let the functional p on m be defined by

$$p((x_i)) = \inf_{i_k} \left\{ \overline{\lim_{t\to\infty}} \frac{1}{n} \left\| \sum_{k=1}^{n} x_{i_k + j} \right\| \right\}, \quad (x_i) \in m,$$

where infimum is taken over all possible choices of natural numbers n and $i_1, i_2, \ldots, i_n$

In a similar fashion as in the proof above we can prove the following statement.

Theorem 2. *There exists a (nonunique) functional L on m with the following properties*:

(i) $L((ax_i+by_i))=aL((x_i))+bL((y_i))\ (a, b\in R,\ (x_i),\ (y_i)\in m)$;

(ii) $L((x_{i+1}))=L((x_i))\ ((x_i)\in m)$;

(iii) $|L((x_i))|\leqslant p((x_i))\ ((x_i)\in m)$.

Also, if $\mathscr{L}$ *denotes the family of all functionals L satisfying the conditions* (i)—(iii), *then for all* $(x_i)\in m$

(iv) $(\forall L\in\mathscr{L})\, L((x_i))=0$ *if and only if* $p((x_i))=0$.

Our next result gives us a connection between the functionals from the theorems 1. $-$2. and Banach limits.

Theorem 3 *Let now E denote the real vector space of all realvalued bounded functions* $t\to f(t)$, $t\geqslant 0$. *Let* $\mathscr{B}$ *be the family of all Banach limits defined on E and* $\mathscr{L}$ *the family of all functionals from the theorem* 1 *defined on E. Then*:

(i) *Every Banach limit is a functional from* $\mathscr{L}$, *i.e.* $B\in\mathscr{B}$ *implies* $B\in\mathscr{L}$,

(ii) *There exists some* $L\in\mathscr{L}$ *such that* $L\notin\mathscr{B}$.

In other words, in the particular case $X=R$, the family $\mathscr{B}$ of all Banach limits is a proper subfamily of the family $\mathscr{L}$ of all functionals from the theorem 1.

Sketch of the proof. Every Banach limit $B\in\mathscr{B}$ is a linear shiftinvariant functional on E, or every Banach limit satisfies the conditions (i)—(ii) from the theorem 1. Also, it is easy to see (see, for example, the Exercise 11.5 from [2]) that $-p(f)\leqslant B(f)\leqslant p(f)$, $f\in E$, $B\in\mathscr{B}$, where p is defined by means of (1), so that the condition (iii) of the theorem 1 is satisfied. Consequently, every Banach limit satisfies the conditions (i)—(iii) of the theorem 1, and we have proved the statement (i) of the theorem.

Last, to complete the proof, consider the realvalued function $t\to f(t)=$ $=s\neq 0$, $t\geqslant 0$. For this function clearly we have $B(f)=s$, $\forall B\in\mathscr{B}$ and $p(f)=|s|>0$, and, as in the proof above, it follows that there exist at least two functionals $L', L''\in\mathscr{L}$ so that $L'(f)\neq L''(f)$. Hence it follows that there exists some $L\in\mathscr{L}$ which is not a Banach limit, i.e. $L\notin\mathscr{B}$. The proof is now complete.

In a similar fashion we can prove the following statement.

Theorem 4. *Let* $1^{+\infty}$ *be the real vector space of all real bounded sequences. Let* $\mathscr{B}$ *denote the family of all Banach limits on* $1^{+\infty}$ *and* $\mathscr{L}$ *the family of all functionals from the theorem* 2 *defined on* $1^{+\infty}$. *Then*:

(i) *Every Banach limit* $B\in\mathscr{B}$ *is a functional from* $\mathscr{L}$.

(ii) *There exists some* $L\in\mathscr{L}$, *which is not a Banach limit, i.e.* $L\notin\mathscr{B}$.

REFERENCES

[1] S. Banach, *Théorie des opérations linéaires*, Warszawa 1932.

[2] G. Bachman and L. Narici, *Functional Analysis*, Academic Press, New York and London 1966.