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IMPLICIT DIFFERENTIAL EQUATION IN LOCALLY CONVEX SPACES

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V. V. Mosyagin and A. I. Povolockij [1] have proved a theorem on the existence of a solution of the equation:

$$(1)\qquad F\left(t, x, \frac{dx}{dt}\right)=0 \quad x(0)=x_0$$

in locally convex spaces, using a generalization of Krasnoseljskij's fixed point theorem which was proved by Millionščikov.

We shall generalize their theorem in two directions, using some results from the theory of ψ-densifying operators and a fixed point theorem which we have proved in [5].

As in [1], let $\{p_i\}_{i\in I}$ be a family of seminorms defining the topology in the locally convex space E,

$$U_b=\{x \mid x\in E,\ p_{i_k}(x-x_0)\leqslant b,\ k=1, 2, \ldots, m; \quad i_k\in I,\ b>0\},$$

$$U_c=\{z \mid z\in E,\ p_{j_r}(z-z_0)\leqslant c,\ r=1, 2, \ldots, s; \quad j_r\in I,\ c>0\}$$

and

$F(t, x, z)=f_0(t, x, z)+f_1(t, x)$ be a mapping from $[0, T]\times U_b\times U_c$ into E. Further, let the mapping f_0 satisfy the following conditions [1]:

1. The mapping f_0 is uniformly continuous on $[0, T]\times U_b\times U_c$ and for every $i\in I$ there exists $L(i)$, $0<L(i)\leqslant 1$ so that:

$$p_i\big(z_1+h_0 f_0(t, x, z_1)-z_2-h_0 f_0(t, x, z_2)\big)\leqslant L(i)\, p_i(z_1-z_2)$$

for every $z_1, z_2\in U_c$, $x\in U_b$, $t\in[0, T]$ $(h_0\neq 0)$

2. The set $\overline{q([0, T], U_b, U_c)}$ is compact, where $g(t, x, z)=z+h_0 f_0(t, x, z)$.

3. $$\sup_{\substack{(t, x, z)\in[0, T]\times U_b\times U_c\\ r=1, \ldots, s}} p_{j_r}\big(g(t, x, z)-z_0\big)+\sup_{\substack{(t, x)\in[0, T]\times U_b\\ r=1, 2, \ldots, s}} p_{j_r}\big(h_0 f_1(t, x)\big)\leqslant c$$

and $T\cdot M_{i_k}\leqslant b$ $\ k=1, 2, \ldots, m$; where

$$\sup_{(t, x, z)\in[0, T]\times U_b\times U_c} p_i\big(g(t, x, z)\big)+\sup_{(t, x)\in[0, T]\times U_b} p_i\big(h_0 f_1(t, x)\big)\leqslant M_i; \text{ for every } i\in I.$$

Now, we shall give some definitions and theorems (see [2])

Definition 1. Let $\mathfrak{M}$ be a subset of 2^E and $Q \in \mathfrak{M}$ implies $\overline{co}\, Q \in \mathfrak{M}$. Further, let $(A, \leqslant)$ be a partially ordered set. The measure of noncompactness ψ is a function $\psi : \mathfrak{M} \to A$ so that $\psi(\overline{co}\, Q) = \psi(Q)$.

Definition 2. $F : \Lambda \times M \to E$, $M \subset E$ is ψ-densifying if the implication $\{\psi[F(\Lambda \times Q)] \geqslant \psi(Q)\} \Rightarrow \{\overline{Q}$ is compact$\}$ holds
The measure ψ is:

a) *monotone* if $Q_1 \subseteq Q_2$ implies $\psi(Q_1) \leqslant \psi(Q_2)$ for every $Q_1, Q_2 \in \mathfrak{M}$.

b) *algebraic semiadditive* if $\psi(Q_1 + Q_2) \leqslant \psi(Q_1) + \psi(Q_2)$ for every Q_1, $Q_2 \in \mathfrak{M}$.

c) *semihomogeneous* if $\psi(zQ) = |z|\, \psi(Q)$ for every $Q \in \mathfrak{M}$ and z-complex number.

d) *continuous* if for every $Q \in \mathfrak{M}$, $p \in P$, $\varepsilon > 0$ there exists a neighbourhood V of the origin so that:

$$|\psi(Q_1)(p) - \psi(Q_2)(p)| < \varepsilon$$

where Q_1 and Q_2 are such that $Q_1 \subseteq Q_2 + V$ and $Q_2 \subseteq Q_1 + V$.

e) *1-regular* if $\psi(Q) = 0$ implies that the set Q is totally bounded.

f) *2-regular* if for every totally bounded set Q the equality $\psi(Q) = 0$ holds.
For example, Kuratowski's measure of noncompactness has all these properties.

Theorem A [2]. *Let R be a closed θ and convex subset of a locally convex space E, let ψ be a measure of noncompactness defined on E and F be b a ψ-densifying mapping of R into R. Suppose that one of the following two conditions is satisfied:*

I. *For every* $(x_0 \in R,\ Q \subseteq R,\ Q \neq \emptyset)$ $\psi(\{x_0\} \cup Q = \psi(Q)$.

II. *ψ is semiadditive and invariant in respect to translation.*

Then $\gamma(\Phi, R) = 1$.

Theorem B (*Theorem* 2 *in* [5]) *Let G be a closed and convex subset of the topological Hausdorff locally convex, complete space E and S, T two mappings of G into E satisfying following conditions:*

1. *For every* $x, y \in G$, $Tx + Sy \in G$.

2. a) *For every $i \in I$ there exists $q(i) \geqslant 0$ and $f : I \to I$ such that*

$$p_i(Tx - Ty) \leqslant q(i)\, p_{f(i)}(x - y) \text{ for every } x, y \in G$$

b) *For every $i \in I$ and $n \in N$ there exist $a_n(i) \geqslant 0$ and $g(i) \in I$ such that for every $x \in E$, $n \in N$ the inequality $p_{f^n(i)}(x) \leqslant a_n(i)\, p_{g(i)}(x)$ holds.*

c) The series

$$\sum_{n=1}^{\infty}\left(\prod_{k=0}^{n-2} q[f^k(i)]\, a_{n-1}(i)\right)$$

is convergent.

3. *The mapping S is continuous and SF is relatively compact set. Then there exists at least one point $x_0 \in G$ such that:*

$$Sx_0 + Tx_0 = x_0.$$

Theorem 1. *Let ψ be a measure of noncompactness defined on E which is invariant in respect to translation and has the properties a) –f). Further, suppose that the mapping $f_1: [0, T] \times U_b \to E$ is uniformly continuous and satisfies the following conditions:*

(*) *For every bounded set $M \subseteq U_b$ and for every $t \in [0, T]$*

$$\psi(f_1(t, M)) \leqslant L(t, \psi(M))$$

where L is a mapping from $[0, T] \times [0, \infty) \to [0, \infty)$ so that the problem $z' = L(t, z)$; $z(0) = 0$ has one and only one solution $z(t) \equiv 0$, $t \in [0, T)$.

If the mappings f_1 and f_0 satisfy conditions 1. — 3. *and if either condition* I *or* II *is satisfied there exists at least one solution of the equation* (1) *which is defined on* $[0, T]$.

Proof: Let $\mathscr{C}^1([0, T], E)$ be the set of all continuously differentiable mapping from $[0, T]$ into E. The topology in $\mathscr{C}^1([0, T], E)$ is defined by the family of the seminorms:

$$\tilde{p}_i(\tilde{x}) = \sup_{t \in [0, T]} p_i[x(t)] + \sup_{t \in [0, T]} p_i[x'(t)]$$

It is known that $\mathscr{C}^1([0, T], E)$ is, in this topology, a complete locally convex space. The measure of noncompactness $\psi_{\mathscr{C}^1}$ on $\mathscr{C}^1([0, T], E)$ is defined in the following way. Let $\mathfrak{M}_{\mathscr{C}^1}$ be a family of all bounded sets $Q \subset \mathscr{C}^1([0, T], E)$ for which the set $Q' = \{\tilde{x}'(t) \mid \tilde{x} \in Q\}$ is equicontinuous. The measure $\psi_{\mathscr{C}^1}$ is introduced by: $\psi_{\mathscr{C}^1}(Q)(t) = \psi(Q_t')$ and maps $\mathfrak{M}_{\mathscr{C}^1}$ into $\mathscr{C}([0, T], E)$. $\mathscr{C}([0, T], E)$ is partially ordered by the relation $\leqslant : f_1 \leqslant f_2 \Leftrightarrow f_1(t) \leqslant f_2(t)\ \forall t \in [0, T]$.

Let

$$V_{1,i} = \{\tilde{x} \mid \tilde{x} \in \mathscr{C}^1([0, T], E),\ x(0) = x_0,\ p_i(x(t_1) - x(t_2)) \leqslant M_i |t_1 - t_2| \text{ for every } (t_1, t_2) \in [0, T]^2\} \quad i \in I$$

$$V_{2,j_r} = \{\tilde{x},\ \tilde{x} \in \mathscr{C}^1([0, T], E),\ p_{j_r}(x'(t) - z_0) \leqslant c, \text{ for every } t \in [0, T]\} \qquad r = 1, 2, \ldots, s.$$

$$V_{3,i} = \left\{x,\ x \in \mathscr{C}^1([0, T], E),\ p_i(x'(t_1) - x'(t_2)) \leqslant \frac{\varphi_i(|t_1 - t_2|)}{1 - L(i)}\right\} \quad i \in I$$

where:

$$\varphi_i(\eta) = \sup_{\tilde{x} \in \bigcap_{i \in I} V_{1,i} \cap (\bigcap_{r=1}^{s} V_{2,j_r}) \; |t_1 - t_2| \leqslant \eta} p_i(g(t_1, x(t_1), x'(t_1)) - g(t_2, x(t_2), x'(t_1))) +$$

$$+ \sup_{\tilde{x} \in \bigcap_{i \in I} V_{1,i} \cap (\bigcap_{r=1}^{s} V_{2,j_r}) \; |t_1 - t_2| \leqslant \eta} p_i\big(h_0 f_1(t_1, x(t_1)) - h_0 f_1(t_2, x(t_2))\big).$$

In is evident that $\varphi_i(\eta) \to 0$ when $\eta \to 0$. In [1] it has been shown that

$$V = \bigcap_{i \in I} V_{1,i} \cap (\bigcap_{r=1}^{s} V_{2,j_r}) \cap (\bigcap_{i \in I} V_{3,i})$$

is closed and convex subset of $\mathscr{C}^1([0, T]), E.)$

The equation (1) is equivalent to the integral equation:

$$x(t) = x_0 + \int_0^t g[s, x(s), x'(s)]\, ds + \int_0^t h_0 f_1[s, x(s)]\, ds \tag{2}$$

The mappings T and S will be defined as follows:

$$Tx = h_0 \int_0^t f_1[s, x(s)]\, ds$$

$$Sx = x_0 + \int_0^t g[s, x(s). \; x'(s)]\, ds.$$

Our aim is to prove that the mapping $T+S$, the set V and the measure $\psi_{\mathscr{C}^1}$ satisfy the conditions of the theorem A. As it has been shown in [3], $T+S$ maps V into V. Next we shall show that the set V is a bounded subset of $\mathscr{C}^1([0, T], E)$. It follows:

$$p_i(x(t)) = p_i(x(t) - x(0) + x(0)) \leqslant p_i(x(t) - x(0)) + p_i(x(0)) \leqslant$$

$\leqslant M_i \cdot |t| + p_i(x_0) \leqslant M_i \cdot T + p_i(x_0)$ for every $\tilde{x} \in V$, since $V \subset \bigcap_{i \in I} V_{1,i}$. Further

$$p_i\left[\frac{x(t+h) - x(t)}{h}\right] \leqslant M_i.$$

and so, by continuity of the seminorms p_i,

$$p_i(x'(t)) = p_i\left[\lim_{h \to 0} \frac{x(t+h) - x(t)}{h}\right] = \lim_{h \to 0} p_i\left[\frac{x(t+h) - x(t)}{h}\right] \leqslant M_i.$$

This means that the set V is a bounded subset of $\mathscr{C}^1([0, T], E)$. From $V \subset \bigcap_{i \in I} V_{3,i}$ it follows that the set $V' = \{x'(t), \; x \in V\}$ is equicontinuous and from $V_0 = \{x(0) \mid x \in V\} = x_0$ that the set V_0 is compact. Using these facts and the condition (*) we conclude, by theorem 1.4.2*[2], that the *mapping T is* $\psi_{\mathscr{C}^1}$-densifying on the set V. Further the set $\overline{g([0, T], U_b, U_c)}$ is compact and the measure ψ is 2-regular, so the mapping S is $\psi_{\mathscr{C}^1}$-densifying on the V. From this we conclude that the mapping $T+S$ is also $\psi_{\mathscr{C}^1}$-densifying on the set $V\subset\mathfrak{M}_{\mathscr{C}^1}$. Now it is easy to show that all conditions of the theorem A are satisfied and that therefore exists at least one solution of the equation (2), i.e. (1).

In [1] f_1 is a contraction type mapping so our theorem generalizes the result of Mosyagin and Povolockij.

Theorem 2. *Suppose that the conditions* 1., 2. *and* 3. *are satisfied and that the mapping f_1 satisfies the following conditions:*

i) *For every $i\in I$, there exist $q(i)\geq 0$ and the mapping $\varphi: I\to I$ so that*

$$p_i(f_1(t, x)-f_1(t, y))\leq q(i)\,p_{\varphi(i)}(x-y)$$

for every $t\in[0, T]$ and $x,y\in U_b$

ii) *For every $i\in I$ and $n\in N$ there exist*

$a_n(i)\geq 0$ and $g: I\to I$ so that:

$p_{\varphi^n(i)}(x)\leq a_n(i)\,p_{g(i)}(x)$ for every $x\in E$ and

$$R=\sup_{i\in I}\ \overline{\lim_{n\in N}}\ \sqrt[n]{\prod_{k=0}^{n-1} q[\varphi^k(i)]\,a_n(i)}\neq\infty \quad \text{and} \quad \frac{1}{Rh_0}>1.$$

Then there exists at least one solution of the equation (1) *defined on* $[0, T_1]$; $T_1=\min\left(T, \dfrac{1}{Rh_0}-1\right)$.

Proof: Let T, S, V and $\mathscr{C}^1$ are as in the theorem 1. We shall prove that all conditions of the theorem B are satisfied. Indeed we have

$$p_i(\tilde{Tx_1}-\tilde{Tx_2})=h_0\Big\{\sup_{t\in[0, T]} p_i\Big[\int_0^t (f_1(s, x_1(s))-f_1(s, x_2(s)))\,ds\Big]+$$

$$+\sup_{t\in[0,T]} p_i[f_1(t, x_1(t))-f_1(t, x_2(t))]\Big\}\leq h_0(T+1)\,q(i)\sup_{t\in[T,0]} p_{\varphi(i)}(x_1(t)-$$

$$-x_2(t))=h_0(T+1)\,q(i)\,\tilde{p}_{\varphi(i)}(\tilde{x}_1-\tilde{x}_2)=Q(i)\,\tilde{p}_{\varphi(i)}(\tilde{x}_1-\tilde{x}_2) \quad \text{and}$$

$$\tilde{p}_{\varphi^n(i)}(\tilde{x})=\sup_{t\in[0, T]} p_{\varphi^n(i)}(x(t))+\sup_{t\in[0, T]} p_{\varphi^n(i)}(x'(t))\leq$$

$$\leq a_n(i)\sup_{t\in[0, T]} p_{g(i)}(x(t))+a_n(i)\sup_{t\in[0, T]} p_{g(i)}(x'(t))=a_n(i)\,\tilde{p}_{g(i)}(\tilde{x})$$

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* In fact this theorem was proved for Banach spaces but it can be easily proved also in the case of locally convex spaces.

Further the series

$$\sum_{n=2}^{\infty}\left(\prod_{k=0}^{n-2} Q\left[\varphi^{k}(i)\right]\right) a_{n}(i)=\sum_{n=2}^{\infty}\left[h_{0}(T+1)\right]^{n-1} \prod_{k=0}^{n-2} q\left[\varphi^{k}(i)\right] \times a_{n-1}(i)$$

is convergent for $T<\frac{1}{Rh_0}-1$.

The conditions 1. and 4. can be easily verified as in [2]. For $\varphi(i)=i$ for every $i\in I$ from our theorem follows the result in [1].

In the special case when $f_1(t, x)=A(t)x$ we can replace the conditions i) and ii) by the following:

For every $i\in I$ and $k\in N$ there exist $q_i(k)\geqslant 0$ and $\varphi: I\to I$, independent from k so that

$$p_i\left(A(t_1) A(t_2)\ldots A(t_k) x-A(t_1) A(t_2)\ldots A(t_k) y\right)\leqslant q_i(k)\, p_{\varphi(i)}(x-y)$$

for every $x, y\in U_b$ and

$$(t_1, t_2, \ldots, t_k)\in \underbrace{[0, T]\times[0, T]\times\cdots\times[0, T]}_{k\text{ — times}} \quad \text{so that}$$

$$0\leqslant t_k\leqslant t_{k-1}\leqslant\cdots\leqslant t_1\leqslant T.$$

$$R=\sup_{i\in I}\ \overline{\lim_{n\in N}}\sqrt[n]{\frac{q_i(n)}{n!}}\neq\infty\left\{h=\min\left(T_1, \frac{1}{Rh_0}\right)\right\}$$

For the proof see [4].

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