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A NOTE ON GENERALIZED TRICOMI POLYNOMIALS

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Abstract

In this note, we have established certain properties of the generalized Tricomi polynomials [3]

$$l_n^\alpha(x)=\sum_{r=0}^{n}(-)^r\binom{x-\alpha}{r}\frac{x^{n-r}}{(n-r)!},$$

with the help of the difference operators.

1. Introduction

We shall use the following formulae in our analysis,

(1.1) $$\Delta_\alpha f(\alpha)=f(\alpha+1)-f(\alpha),$$

(1.2) $$E_\alpha f(\alpha)=f(\alpha+1),$$

(1.3) $$\Delta_\alpha^n[u_\alpha v_\alpha]=\sum_{r=0}^{n}\binom{n}{r}\Delta_\alpha^{n-r}u_{\alpha+r}\Delta_\alpha^r v_\alpha,$$

(1.4) $$\Delta_\alpha^n[x^\alpha u_\alpha[=x^\alpha[xE-1]^n u_\alpha,$$

and

(1.5) $$\Delta_\alpha^n f(\alpha)=\sum_{r=0}^{n}(-)^{n-r}\binom{n}{r}f(\alpha+r).$$

2. Starting with

$$\frac{(-x)^{n+\alpha}}{n!\,\Gamma(\alpha-x)}\Delta_\alpha^n[(-x)^{-\alpha}\Gamma(\alpha-x)]$$

and using (1.5), we get

(2.1) $$l_n^{\alpha}(x)=\frac{(-x)^{n+\alpha}}{n!\,\Gamma(\alpha-x)}\Delta_{\alpha}^{n}[(-x)^{-\alpha}\Gamma(\alpha-x)].$$

The use of (1.1), will give

(2.2) $$\Delta_{\alpha}\, l_n^{\alpha}(x)=l_{n-1}^{\alpha+1}(x).$$

Hence by iteration, we have

(2.3) $$\Delta_{\alpha}^{r}\, l_n^{\alpha}(x)=l_{n-r}^{\alpha+r}(x).$$

3. In this section we shall prove,

(3.1) $$n!\sum_{r=0}^{n}\frac{(\alpha-x)_r}{r!}\, l_{n-r}^{\alpha+r}(x)\,\Delta_{\alpha}^{r}$$

$$=\frac{x^n}{\Gamma(\alpha-x)}[1+x^{-1}E]^n\Gamma(\alpha-x)=\prod_{j=1}^{n}[(\alpha-x-n+2j+1)E+x].$$

Let

(3.2) $$\Omega_n f(\alpha)=\frac{(-x)^{n+\alpha}}{\Gamma(\alpha-x)}\Delta_{\alpha}^{n}[(-x)^{\alpha}\Gamma(\alpha-x)f(\alpha)].$$

Using (1.3), we have

$$\Omega_n f(\alpha)=\frac{(-x)^{n+\alpha}}{\Gamma(\alpha-x)}\sum_{r=0}^{n}\binom{n}{r}\Delta_{\alpha}^{n-r}(-x)^{-\alpha-r}\Gamma(\alpha+r-x)\,\Delta_{\alpha}^{r}f(\alpha),$$

now, with the help of (2.1), we establish

(3.3) $$\Omega_n f(\alpha)=n!\sum_{r=0}^{n}\frac{(\alpha-x)_r}{r!}\, l_{n-r}^{\alpha+r}(x)\,\Delta_{\alpha}^{r}f(\alpha).$$

Using (1.4), the equation (3.2) can be put into the following form

(3.4) $$\Omega_n f(\alpha)=\frac{x^n}{\Gamma(\alpha-x)}[1+x^{-1}E]^n\,\Gamma(\alpha-x)f(\alpha).$$

Again starting from (3.2), and using (1.3), we have

$$\Omega_n f(\alpha)=\frac{(-x)^{n+\alpha}}{\Gamma(\alpha-x)}[(\alpha-x-1)\Delta\{\Delta^{n-1}(-x)^{-\alpha}\Gamma(\alpha-x-1)f(\alpha)\}$$

$$+nE\{\Delta^{n-1}(-x)^{-\alpha}\Gamma(\alpha-x-1)f(\alpha)\}]$$

$$=\frac{(-x)^{n+\alpha}}{\Gamma(\alpha-x)}[(\alpha-x-1)\Delta+nE]\{\Delta^{n-1}(-x)^{-\alpha}\Gamma(\alpha-x-1)f(\alpha)\}$$

$$=\frac{(-x)^{n+\alpha}}{\Gamma(\alpha+x)}[(\alpha-x-1)\Delta+nE]\cdots[(\alpha-x-n)\Delta+1\cdot E](-x)^{-\alpha}\Gamma(\alpha-x-n)f(\alpha).$$

Therefore,

$$\Omega_n f(\alpha) = \prod_{j=1}^{n} [(\alpha - x - n + 2j - 1)\, E + x] f(\alpha). \tag{3.5}$$

From (3.3), (3.4) and (3.5) we get the required result (3.1).

4. Recurrence relations

From (2.2), we have

$$l_n^{\alpha}(x) = l_n^{\alpha-1}(x) + l_{n-1}^{\alpha}(x). \tag{4.1}$$

Again starting from (2.1), it can be shown that

$$(n+1)\, l_{n+1}^{\alpha}(x) = (\alpha - x)\, l_n^{\alpha+1}(x) + x\, l_n^{\alpha}(x). \tag{4.2}$$

For $f(\alpha) = 1$, (3.1) will give

$$n!\, l_n^{\alpha}(x) = \prod_{j=1}^{n} [(\alpha - x - n + 2j - 1)\, E + x]\, 1. \tag{4.3}$$

Now consider

$$(n+1)!\, l_{n+1}^{\alpha}(x) = \prod_{j=1}^{n+1} [(\alpha - x - n + 2j - 2)\, E + x]\, 1$$

which after some simplification will give

$$(n+1)\, l_{n+1}^{\alpha}(x) = (n + \alpha - x)\, l_n^{\alpha}(x) + x\, l_n^{\alpha-1}(x). \tag{4.4}$$

From (4.1) and (4.4), we derive

$$(n+1)\, l_{n+1}^{\alpha}(x) = (\alpha + n)\, l_n^{\alpha}(x) - x\, l_{n-1}^{\alpha}(x), \tag{4.5}$$

which is (3.2) in [3].

5. Other relations

Repeated application of (4.1) gives

$$l_{n-k}^{\alpha}(x) = \sum_{r=0}^{k} (-)^r \binom{k}{r}\, l_n^{\alpha-r}. \tag{5.1}$$

Again using the famous result

$$f(\alpha + \mu) = \sum_r \binom{\mu}{r}\, \Delta_\alpha^r f(\alpha),$$

we get

$$l_n^{\alpha+\mu}(x) = \sum_{r=0}^{n} \binom{\mu}{r}\, l_{n-r}^{\alpha+r}(x). \tag{5.2}$$

In (4.3) replacing n by $n+m$ and applying (3.1) and (2.3), we derive

$$\binom{m+n}{m}\, l_{m+n}^{\alpha}(x) = \sum_{r=0}^{m} \frac{(\alpha + n - x)_r}{r!}\, l_{m-r}^{\alpha+n+r}(x)\, l_{n-r}^{\alpha+r-m}(x). \tag{5.3}$$

If we take $f(\alpha)=\dfrac{1}{\Gamma(\alpha-x)}$, then (3.3) and (3.4) will give

$$(5.4)\qquad (1+x)^n=\sum_{r=0}^{n} n!\frac{(-)^r(\alpha-x)_r}{r!}\, l_{n-r}^{\alpha+r}(x)\,{}_1F_1\left[\begin{matrix}-r; \\ \alpha-x;\end{matrix}\ 1\right].$$

Now putting $f(\alpha)=\dfrac{x^\alpha}{\Gamma(\alpha+1)}$, and applying the famous result [4]

$$L_n^\alpha(x)=(-)^n\frac{\Gamma(\alpha+n+1)}{n!}x^{-\alpha}\Delta_\alpha^n\left[\frac{x^\alpha}{\Gamma(\alpha+1)}\right]$$

we observe that,

$$(5.5)\qquad x^n\cdot{}_2F_1\left[\begin{matrix}-n,\alpha-x; \\ \alpha+1;\end{matrix}\ -1\right]=\sum_{r=0}^{n} n!\frac{(-)^r(\alpha-x)_r}{(\alpha+1)_r}\, l_{n-r}^{\alpha+r}(x)\,L_r^\alpha(x).$$

Again putting $f(\alpha)=x^{-\alpha}\Gamma(1+\alpha)$ and taking the help of the well known result [2]

$$A_n^\alpha(x)=\frac{(-)^n x^{n+\alpha}}{n!\,\Gamma(1+\alpha)}\Delta_\alpha^n[x^{-\alpha}\Gamma(1+\alpha)],$$

we get

$$(5.6)\qquad x^n\cdot{}_3F_0\left[\begin{matrix}-n,\alpha-x,1+\alpha; \\ \text{—}\end{matrix}\ -x^{-2}\right]=\sum_{r=0}^{n} n!\,(\alpha-x)_r(-x)^{-r}\, l_{n-r}^{\alpha+r}(x)\,A_r^\alpha(x),$$

where $A_n^\alpha(x)$ is the polynomials defined by Srivastava [6].

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