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THEOREMS ON STRONG SUMMABILITY

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1. Introduction. A sequence $\{s_n\}$ is said to be summable (C, p, β) for $p>-1$, $\beta>-1$ and $p+\beta>-1$, to the sum s, if

$$C^p_{n,\beta}=S^p_{n,\beta}/E^p_{n,\beta}\rightarrow s, \text{ as } n\rightarrow\infty, \tag{1.1}$$

where $E^p_{n,\beta}$ and $S^p_{n,\beta}$ are defined by

$$\sum_{n=0}^{\infty} E^p_{n,\beta}\, x^n=(1-x)^{-(p+\beta+1)} \quad\text{and}\quad \sum_{n=0}^{\infty} S^p_{n,\beta}\, x^n=(1-x)^{-p}\sum_{n=0}^{\infty} E^{\beta}_n s_n x^n.$$

Here

$$E^p_{n,\beta}=\sum_{r=0}^{n} E^{p-1}_{r,\beta},\quad S^p_{n,\beta}=\sum_{r=0}^{n} S^{p-1}_{r,\beta},\quad E^p_{n,\beta}=E^{p+\beta}_n=\binom{n+p+\beta}{n}\sim\frac{n^{p+\beta}}{\Gamma(p+\beta+1)},$$

$$E^{p+\delta}_{n,\beta}=\sum_{r=0}^{n} E^{\delta-1}_{n-r}E^p_{r,\beta},\quad S^{p+\delta}_{n,\beta}=\sum_{r=0}^{n} E^{\delta-1}_{n-r}S^p_{r,\beta}$$

and

$$C^{p+\delta}_{n,\beta}=\frac{1}{E^{p+\delta}_{n,\beta}}\sum_{r=0}^{n} E^{\delta-1}_{n-r}E^p_{r,\beta}\,C^p_{r,\beta}\qquad(\text{for } \delta>0).$$

When $p>0$, (C, p, β) method defined by (1.1) is a regular Nörlund method and $\{C^p_{n,\beta}\}$ is a regular Hausdorff transform of the sequence $\{s_n)$ generated by the sequence $\left\{\frac{E^{\beta}_n}{E^p_{n,\beta}}=\frac{\Gamma(p+\beta+1)\,\Gamma(n+\beta+1)}{\Gamma(\beta+1)\,\Gamma(n+p+\beta+1)}\right\}$. If we put $\beta=0$ in the (C, p, β) method, we get the familiar Cesàro method (C, p) of order $p>-1$. A. Zygmund [9] proved that the methods (C, p, β) and (C, p) are equivalent. Summability (C, O, β) means the convergence.

For $p>-1$, $\beta>-1$ and $p+\beta>-1$ let $T^p_{n,\beta}=t^p_{n,\beta}/E^p_{n,\beta}$ and

$$\sum_{n=1}^{\infty} t^p_{n,\beta}\, x^n=(1-x)^{-p}\sum_{n=1}^{\infty} E^{\beta}_n n\,(s_n-s_{n-1})\, x^n=\sum_{n=0}^{\infty} E^{p-1}_n x^n\sum_{n=1}^{\infty} t^0_{n,\beta}\, x^n. \tag{1.2}$$

Here

$$t^{0}_{n,\beta}=E^{\beta}_{n}n(s_n-s_{n-1}),\quad t^{p}_{n,\beta}=\sum_{r=1}^{n}E^{p-1}_{n-r}\,t^{0}_{r,\beta}\,,\quad t^{p+\delta}_{n,\beta}=\sum_{r=1}^{n}E^{\delta-1}_{n-r}\,t^{p}_{r,\beta}$$

and

$$(1.3)\qquad T^{p+\delta}_{n,\beta}=\frac{1}{E^{p+\delta}_{n,\beta}}\sum_{r=1}^{n}E^{\delta-1}_{n-r}E^{p}_{r,\beta}\,T^{p}_{r,\beta}\qquad(\text{for }\delta>0).$$

Let $f(x)=\sum_{n=0}^{\infty}E^{\beta}_{n}s_n x^n$. Then $\sum_{n=0}^{\infty}S^{p}_{n,\beta}\,x^n=(1-x)^{-p}f(x)$. Hence

$$(1.4)\qquad \sum_{n=0}^{\infty}(p+\beta)\,S^{p}_{n,\beta}\,x^n=(p+\beta)\,(1-x)^{-p}f(x)$$

and

$$\sum_{n=0}^{\infty}S^{p-1}_{n,\beta}\,x^{p+\beta+n}=x^{p+\beta}(1-x)^{-(p-1)}f(x).$$

Therefore

$$(1.5)\qquad \sum_{n=0}^{\infty}(p+\beta+n)\,S^{p-1}_{n,\beta}\,x^n=x^{-(p+\beta-1)}\frac{d}{dx}\{x^{p+\beta}(1-x)^{-(p-1)}f(x)\}.$$

From (1.4) and (1.5) we get

$$(1.6)\qquad \sum_{n=0}^{\infty}(p+\beta+n)\,S^{p-1}_{n,\beta}\,x^n-\sum_{n=0}^{\infty}(p+\beta)\,S^{p}_{n,\beta}\,x^n$$

$$=-(\beta+1)(1-x)^{-p}xf(x)+x(1-x)^{-(p-1)}f'(x)$$

$$=(1-x)^{-p}\sum_{n=1}^{\infty}E^{\beta}_{n}n(s_n-s_{n-1})\,x^n=\sum_{n=1}^{\infty}t^{p}_{n,\beta}\,x^n\quad\text{by (1.2).}$$

From (1.6) we get $t^{p}_{n,\beta}=(p+\beta+n)\,S^{p-1}_{n,\beta}-(p+\beta)\,S^{p}_{n,\beta}$ and hence

$$(1.7)\qquad T^{p}_{n,\beta}=(p+\beta)\left(C^{n-1}_{n,\beta}-C^{p}_{n,\beta}\right)=n\left(C^{p}_{n,\beta}-C^{p}_{n-1,\beta}\right).$$

A sequence $\{s_n)$ is said to be summable by the generalized Abel method (A_α), for a real number $\alpha>-1$, to the sum s, if $\sum_{n=0}^{\infty}E^{\alpha}_{n}s_n x^n$ is convergent for all x in $0\leqslant x<1$ and $A_\alpha(x)=(1-x)^{\alpha+1}\sum_{n=0}^{\infty}E^{\alpha}_{n}s^n x^n\to s$, as $x\to 1-0$. This is briefly denoted by $s_n\to s\,(A_\alpha)$. This method was introduced independently by A. Amir Jakimovski ([1] p. 374) and C. T. Rajagopal ([8] p. 93). The properties of this method were discussed in detail by D. Borwein [2]. In the sequence to function transformation method (A_α) if we put $\alpha=0$, we get the familiar Abel method (A_0) or (A).

A sequence $\{s_n\}$ is said to be summable by the generalized Abel $-(C, p, \beta)$ method $(A_\alpha;\ C, p, \beta)$, for $p>-1$, $\alpha>-1$, $\beta>-1$ and $p+\beta>-1$, to the sum s, if $\sum_{n=0}^{\infty} E_n^\alpha C_{n,\beta}^p x^n$ is convergent for all x in $0\leqslant x<1$ and

$$(1.8)\qquad A_\alpha^{(p,\beta)}(x)=(1-x)^{\alpha+1}\sum_{n=0}^{\infty} E_n^\alpha C_{n,\beta}^p x^n\to s,\ \text{as}\ x\to 1-0,\ \text{i.e.}\ C_{n,\beta}^p\to s\,(A_\alpha).$$

The sequence to function transformation method $(A_\alpha;\ C, p, \beta)$ reduces to

(i) generalized Abel-Cesàro method $(A_\alpha;\ C, p)$ when $\beta=0$,

(ii) Abel $-(C, p, \beta)$ method $(A;\ C, \text{p}, \beta)$ when $\alpha=0$,

(iii) familiar Abel-Cesàro method $(A;\ C, p)$ when $\alpha=0$ and $\beta=0$,

(iv) generalized Abel method (A_α) when $p=0$ and

(v) familar Abel method (A) when $p=0$ and $\alpha=0$.

In this paper strong summability methods $\{C, p, \beta\}_k$, $[C, p, \beta]_k$, $\{A_\alpha;\ C, p, \beta\}_k$ *and* $[A_\alpha;\ C, p, \beta]_k$ *based upon summability methods* (C, p, β) *and* $(A_\alpha;\ C, p, \beta)$ *are defined, and various implications between these strong summability methods and the ordinary summability methods* (C, p, β), (C, p) *and* $(A_\alpha;\ C, p, \beta)$ *are investigated as generalizations of the corresponding results due to T. M. Flett* [4] *and B. P. Mishra* [6 *and* 7]. *The 'o' depth and 'O' depth Tauberian theorems for summability methods* $\{C, q, \beta\}_k$ *and* $\{A_\alpha;\ C, q, \beta\}_k$ *with summability and boundedness* $[C, p+1, \beta]_k$ *as Tauberian conditions are established as generalizations of the corresponding results due to T. M. Flett* [4]. *In the latter part the 'o' depth and 'O' depth Tauberian theorems for the ordinary summability method* $(A_\alpha;\ C, q, \beta)$ *with the Tauberian conditions* $(C_{n,\beta}^{p+1}-C_{n-1,\beta}^{p+1})=o\,(n^{-1})$ *and* $(C_{n,\beta}^{p+1}-C_{n-1,\beta}^{p+1})=O\,(n^{-1})$ *are deduced.*

For any number $k>1$ used as an index, we write $k'=k/(k-1)$, so that k and k' are conjugate indices in the sense of Hölder's inequality. $1/k'=0$ when $k=1$.

We use $D(a, b.\ c, \ldots)$ to denote a positive constant depending only on $a, b, c, \ldots.$ not necessarily the same on any two occurrences, D by itself will denote a positive absolute constant.

Inequalities of the form $M\leqslant D(a, b, c, \ldots)N$ are to be interpreted as meaning "if the expression N is finite, then the expression M is also finite and satisfies the inequality".

2. Strong Summability. A sequence $\{s_n\}$ is said to be strongly summable $(C, p+1, \beta)$ with index k, or summable $\{C, p, \beta\}_k$ to the sum s for $p>-1$, $\beta>-1$, $p+\beta>-1$ and $k\geqslant 1$, if

$$(2.1)\qquad (N+1)^{-1}\sum_{n=0}^{N}\left|C_{n,\beta}^p-s\right|^k=o\,(1),\ \text{as}\ N\to\infty.$$

If (2.1) is true for some $p>-1$, then $\{s_n\}$ is said to be summable $\{C, *, \beta\}_k$ to s. The sequence $\{s_n\}$ is said to be bounded $\{C, p, \beta\}_k$ if

$$(2.2)\qquad (N+1)^{-1}\sum_{n=0}^{N}\left|C_{n,\beta}^p\right|^k=O\,(1),\ \text{as}\ N\to\infty.$$

If (2.2) is true for some $p>-1$, then $\{s_n\}$ is said to be bounded $\{C, *, \beta\}_k$.

A sequence $\{s_n\}$ is said to be strongly summable $(A_\alpha;\ C, p, \beta)$ with index k, or summable $\{A_\alpha;\ C, p, \beta\}_k$, to the sum s for $p>-1$, $\alpha>-1$, $\beta>-1$, $p+\beta>-1$ and $k\geqslant 1$, if the series $\sum_{n=0}^{\infty} E_n^\alpha C_{n,\beta}^p x^n$ is convergent for all x in $0\leqslant x<1$ and $A_\alpha^{(p,\beta)}(x)$ defined by (1.8) satisfies the condition

$$(2.3)\qquad (1-R)\int_0^R \left| A_\alpha^{(p,\beta)}(x)-s\right|^k (1-x)^{-2}\,dx=o(1), \text{ as } R\to 1-0.$$

The sequence $\{s_n\}$ is said to be bounded $\{A_\alpha;\ C, p, \beta\}_k$ if

$$(2.4)\qquad (1-R)\int_0^R \left| A_\alpha^{(p,\beta)}(x)\right|^k (1-x)^{-2}\,dx=O(1), \text{ as } R\to 1-0.$$

Summability $\{C, p, \beta\}_k$ to the sum s is equivalent to $\left|C_{n,\beta}^p-s\right|^k\to 0\ (C, 1)$, as $n\to\infty$ and summability $\{A_\alpha;\ C, p, \beta\}_k$ to the sum s is equivalent to $\left|A_\alpha^{(p,\beta)}(x)-s\right|^k\to 0\ (C, 1)$, as $x\to 1-0$. This follows by integration by parts.

Let $\gamma>1$ be a fixed number. Then condition (2.1) is equivalent to

$$(2.5)\qquad \left\{N^{\gamma-1}\sum_{n=N}^{\infty}\left|C_{n,\beta}^p-s\right|^k (n+1)^{-\gamma}\right\}^{1/k}=o(1).$$

As $k\to\infty$ the expression on the left of (2.5) tends to $\sup_{n\geqslant N}\left|C_{n,\beta}^p-s\right|$, so that the limiting form of (2.5) as $k\to\infty$ is that $C_{n,\beta}^p-s=o(1)$, as $n\to\infty$. Thus summability (C, p, β) may be regarded as the case $k=\infty$ of summablity $\{C, p, \beta\}_k$.

It is necessary to transform (2.1) into (2.5) in order to obtain a reasonable definition of summability $\{C, p, \beta\}_k$ for $k=\infty$. If we take the $(1/k)^{th}$ power of both sides of (2.1) and make $k\to\infty$, we obtain formally $\sup_{n\leqslant N}\left|C_{n,\beta}^p-s\right|=o(1)$, and this implies that $C_{n,\beta}^p=s$ for all n. Boundedness (C, p, β) may be regarded as the case $k=\infty$ of boundedness $\{C, p, \beta\}_k$.

Similarly summability $(A_\alpha;\ C, p, \beta)$ and boundedness $(A_\alpha;\ C, p, \beta)$ may be regarded as the case $k=\infty$ of summability $\{A_\alpha;\ C, p, \beta\}_k$ and boundedness $\{A_\alpha;\ C, p, \beta\}_k$ respectively.

We shall now define strong summability methods involving the expression $T_{n,\beta}^p$. A sequence $\{s_n\}$ is said to be summable $[C, p, \beta]_k$, for $p>-1$, $\alpha>-1$, $p+\beta>-1$ and $k\geqslant 1$, if

$$(2.6)\qquad (N+1)^{-1}\sum_{n=1}^{N}\left|T_{n,\beta}^p\right|^k=o(1), \text{ as } N\to\infty.$$

We may also regard the condition

$$(2.7)\qquad T_{n,\beta}^p=o(1), \text{ as } n\to\infty$$

as the case $k=\infty$ of the summability $[C, p, \beta]_k$. The sequence $\{s_n\}$ is said ot be bounded $[C, p, \beta]_k$ if (2.6) or (2.7) holds with o replaced by O. If (2.6) or (2.7) is true for some $p>-1$, then the sequence $\{s_n\}$ is said to be summable $[C, *, \beta]_k$, and similarly in the case of boundedness.

Corresponding to summability $[C, p\ \beta]_k$ we have the generalized Abel-(C, p, β) summability $[A_\alpha;\ C, p, \beta]_k$ defined as follows. A sequence $\{s_n\}$ is said to be summable $[A_\alpha;\ C, p, \beta]_k$, for $p > -1$, $\alpha > -1$, $\beta > -1$, $p + \beta > -1$ and $k \geqslant 1$, if

$$(2.8)\qquad (1-R)\int_0^R (1-x)^{k-2}\left|A_\alpha^{(p,\beta)}(x)\right|^k dx = o(1), \text{ as } R \to 1-0.$$

When $k = \infty$, the condition (2.8) is being replaced by

$$(2.9)\qquad (1-x)\, A_\alpha^{(p,\beta)'}(x) = o(1), \text{ as } x \to 1-0.$$

The sequence $\{s_n\}$ is said to be bounded $[A_\alpha;\ C, p, \beta]_k$, if (2.8) or (2.9) holds with o replaced by O.

When $\beta = 0$, summability methods $\{C, p, \beta\}_k$ and $[C, p, \beta]_k$ reduce respectively to the summability methods $\{C, p\}_k$ and $\{c, p\}_k$ defined by T. M. Flett [4].

When $p = 0$, summability method $\{A_\alpha;\ C, p, \beta\}_k$ reduces to summability method $\{A_\alpha\}_k$. This definition of summability $\{A_\alpha\}_k$ is equivalent to the definition given by B. P. Mishra [7], being obtained by obvious changes of variable and parameter. When $\alpha = 0$ and $p = 0$ summability method $\{A_\alpha;\ C, p, \beta\}_k$ reduces to summability method $\{A\}_k$ defined by T. M. Flett [4]. When $\alpha = 0$ and $\beta = 0$ summability method $\{A_\alpha;\ C, p, \beta\}_k$ reduces to summability method $\{A;\ C, p\}_k$. This definition of summability $\{A;\ C, p\}_k$ is equivalent to the definition given by B P. Mishra [6], being obtained by obvious changes of variable and parameter.

When $p = 0$, summability method $[A_\alpha;\ C, p, \beta]_k$ reduces to summability method $[A_\alpha]_k$. The condition to be satisfied by $\{s_n\}$ for $[A_\alpha]_k$ summability is equivalent to the condition imposed in the known result of B. P. Mishra ([7] p. 122, Theorem 4), being obtained by obvious changes of variable and parameter. When $\alpha = 0$ and $p = 0$ summability method $[A_\alpha;\ C, p, \beta]_k$ reduces to summability method $[A]_k$ defined by T. M. Flett [4], and denoted by him as $\{A\}_k$. When $\alpha = 0$ and $\beta = 0$ summability method $[A_\alpha;\ C, p, \beta]_k$ reduces to summability method $[A;\ C, p]_k$. This definition of summability $[A;\ C, p]_k$ is equivalent to the definition given by B. P. Mishra [6], being obtained by obvious changes of variable and parameter.

3. Theorems. We shall establish the implications between the summability methods defined above.

3.1. Theorem 1. (i) Let $p > -1$, $\beta > -1$, $p + \beta > -1$ and $1 \leqslant k \leqslant \infty$. If a sequence $\{s_n\}$ is summable $\{C, p, \beta\}_k$ to the sum s, then it is summable $\{C, p, \beta\}_m$ to the same sum s for every m such that $1 \leqslant m \leqslant k$.

(ii) Let $p > -1$, $\alpha > -1$, $\beta > -1$, $p + \beta > -1$ and $1 \leqslant k \leqslant \infty$. If a sequence $\{s_n\}$ is summable $\{A_\alpha;\ C, p, \beta\}_k$ to the sum s, then it is summable $\{A_\alpha;\ C, p, \beta\}_m$ to the same sum s for every m such that $1 \leqslant m \leqslant k$.

(iii) If $p > -1$, $\alpha > -1$, $\beta > -1$, $p + \beta > -1$ and $0 < m < k < \infty$, then for any s

$$(3.1.1)\qquad \left\{(N+1)^{-1}\sum_{n=0}^{N}\left|C_{n,\beta}^{p} - s\right|^m\right\}^{1/m} \leqslant \left\{(N+1)^{-1}\sum_{n=0}^{N}\left|C_{n,\beta}^{p} - s\right|^k\right\}^{1/k} \leqslant \operatorname*{Sup}_{n \leqslant N}\left|C_{n,\beta}^{p} - s\right|$$

and

$$(3.1.2)\qquad \left\{(1-R)\int_0^R \left|A_\alpha^{(p,\beta)}(x)-s\right|^m (1-x)^{-2}\,dx\right\}^{1/m}$$

$$\leq \left\{(1-R)\int_0^R \left|A_\alpha^{(p,\beta)}(x)-s\right|^k (1-x)^{-2}\,dx\right\}^{1/k}$$

$$\leq \operatorname{Sup}_x \left|A_\alpha^{(p,\beta)}(x)-s\right| \leq \sup_n \left|C_{n,\beta}^p - s\right|.$$

(iv) Throughout (i)—(iii) we may replace $\{C, p, \beta\}$ by $[C, p, \beta]$, $\{A_\alpha; C, p, \beta\}$ by $[A_\alpha; C, p, \beta]$ (with omission of the sum s), $(C_{n,\beta}^p - s)$ by $T_{n,\beta}^p$ and $A_\alpha^{(p,\beta)}(x)-s$ by $(1-x)\,A_\alpha^{(p,\beta)\prime}(x)$.

The first inequalities in (3.1.1) and (3.1.2) follow from Hölder's inequality. The second inequalities in (3.1.2) and (3.1.1) are obvious and the third inequality in (3.1.2) follows from the identity

$$A_\alpha^{(p,\beta)}(x) = (1-x)^{\alpha+1}\sum_{n=0}^{\infty} E_n^\alpha C_{n,\beta}^p x^n \text{ for } 0 \leq x < 1,$$

since $(1-x)^{\alpha+1}\sum_{n=0}^{\infty} E_n^\alpha x^n = 1$. The inequalities (3.1.1) and (3.1.2) are analogues of (i) and (ii) for boundedness $\{C, p, \beta\}_k$ and $\{A_\alpha; C, p, \beta\}_k$.

Theorem 1 is a collection of elementary results in the direction of decreasing k. The complicated results in the direction of increasing k are collected together in the following Theorem.

3.2. Theorem 2. Let $p>-1$, $\beta>-1$, $p+\beta>-1$ and either $1<k\leq m<\infty$ and $q\geq p+\frac{1}{k}-\frac{1}{m}$ or $1=k\leq m<\infty$ and $q>p+\frac{1}{k}-\frac{1}{m}$.

(i) If a sequence $\{s_n\}$ is summable $\{C, p, \beta\}_k$ to the sum s, then it is summable $\{C, q, \beta\}_m$ to the same sum s.

(ii) For any s

$$(3.2.1)\qquad \operatorname{Sup}_N \left\{(N+1)^{-1}\sum_{n=0}^{N}\left|C_{n,\beta}^q - s\right|^m\right\}^{1/m}$$

$$\leq D(k, m, p, q, \beta)\operatorname{Sup}_N \left\{(N+1)^{-1}\sum_{n=0}^{N}\left|C_{n,\beta}^p - s\right|^k\right\}^{1/k}.$$

(iii) In (i) and (ii) we may replace $\{C, p, \beta\}$ by $[C, p, \beta]$ $(C_{n,\beta}^p - s)$ by $T_{n,\beta}^p$ and $(C_{n,\beta}^q - s)$ by $T_{n,\beta}^q$.

3.3. Theorem 3. *Let* $p>-1$, $\beta>-1$, $p+\beta>-1$ *and either* $k>1$ *and* $q>p^+\ 1/k$ *or* $k=1$ *and* $q\geq p+1$.

(i) If a sequence $\{s_n\}$ is summable $\{C, p, \beta\}_k$ to the sum s, then it is summable (C, q, β) and hence (C, q) to the same sum s.

(ii) For any s

$$\operatorname*{Sup}_{N}\left|C^{q}_{N,\beta}-s\right|\leqslant D(k,p,q,\beta)\operatorname*{Sup}_{N}\left\{(N+1)^{-1}\sum_{n=0}^{N}\left|C^{p}_{n,\beta}-s\right|^{k}\right\}^{1/k}.$$

(iii) If a sequence $\{s_n\}$ is summable $[C, p, \beta]_k$, then $T^{q}_{n,\beta}=o(1)$, as $n\to\infty$.

(iv) $$\operatorname*{Sup}_{N}\left|T^{q}_{N,\beta}\right|\leqslant D(k,p,q,\beta)\operatorname*{Sup}_{N}\left\{(N+1)^{-1}\sum_{n=0}^{N}\left|T^{p}_{n,\beta}\right|^{k}\right\}^{1/k}.$$

Theorems 2 and 3 can be proved by the arguments similar to that of Theorems 2 and 3 of T. M. Flett [4].

Remark. From theorems 1 and 3 we get that a sequence $\{s_n\}$ is summable $\{C, *, \beta\}_k$ if and only if it is summable $(C, *, \beta)$.

3.4. Theorem 4. *Let* $p>-1$, $q\geqslant 0$, $\alpha>-1$, $\beta>-1$, $p+\beta>-1$ *and* $1\leqslant k\leqslant\infty$.

(i) If a sequence $\{s_n\}$ is summable $\{C, p, \beta\}_k$ to the sum s, then it is summable $\{A_\alpha; C, q, \beta\}_m$ to the same sum s for every m $(1\leqslant m\leqslant\infty)$.

(ii) If a sequence $\{s_n\}$ is summable $[C, p, \beta]_k$, then it is summable $[A_\alpha; C, q, \beta]_m$ for every $m\,(1\leqslant m\leqslant\infty)$.

Proof of Theorem 4. If a sequence $\{s_n\}$ is summable $\{C, p, \beta\}_k$ to the sum s, by Theorem 3 (i), it is summable (C, p') where $p'>p+1/k$ if $1<k$ and $p'\geqslant p+1$ if $k=1$, to the same sum s and hence it is summable (A_α) to s by the following Lemma 1.

Lemma 1. (see D. Borwein [2].) *If a sequence* $\{s_n\}$ *is summable* (C, p) $(p>-1)$ *to the sum s, then it is summable* $(A_\alpha)\,(\alpha>-1)$ *to the same sum s.*

Now by the following Lemma 2 we observe that $\{s_n\}$ is summable (A_α, C, q, β) to s, since $\{C^{q}_{n,\beta}\}$ is a regular Hausdorff transform of $\{s_n\}$.

Lemma 2. (see A. Amir Jakimovski [1]). *Let* $\alpha>-1$ *be a real number. If a sequence* $\{s_n\}$ *is summable* (A_α) *to the sum s and* $\{h_n\}$ *is a regular Hausdorff transform of* $\{s_n\}$, *then* $\{h_n\}$ *is summable* (A_α) *to the same sum s.*

Hence by Theorem 1 (ii) with $k=\infty$, we get that the sequence $\{s_n\}$ is summable $\{A_\alpha; C, q, \beta\}_m$ to the same sum s for every m. Hence (i) is proved and (ii) follows from (i) applied to the sequence $\{n(s_n-s_{n-1})\}$.

The special case $q=0$, $\alpha=0$ and $\beta=0$ of this theorem is a known result of T. M. Flett ([4] p. 120, Theorem 4). The known result of D. Borwein ([2] p. 320, Theorem 4) which is used to prove this theorem is a special case of this theorem with $q=0$, $\beta=0$ and $k=m=\infty$. This theorem also includes the known result of B. P. Mishra ([6] p. 316, Theorem 6) as a special case with $\alpha=0$, $\beta=0$ and $q=p$, $k=m$. And further the known result of B. P. Mishra ([7] p. 125, Theorem 5) is a particular case of this theorem with $\beta=0$ and $q=0$.

Theorem 4 establishes the connection between summability $\{C, p, \beta\}_k$ with summability $\{A_\alpha; C, q, \beta\}_m$ and summability $[C, p, \beta]_k$ with summability $[A_\alpha; C, q, \beta]_m$. We have also the following convexity theorem.

3.5. Theorem 5. *Let $q>p>-1$, $\beta>-1$, $p+\beta>-1$ and $1\leqslant k\leqslant\infty$.*

(i) If a sequence $\{s_n\}$ is bounded $\{C, p, \beta\}_k$ and summable $\{C, *, \beta\}_k$ or $(C, *, \beta)$ to the sum s, then it is summable $\{C, q, \beta\}_k$ to s.

(ii) If a sequence $\{s_n\}$ is bounded $[C, p, \beta]_k$ and summable $[C, *, \beta]_k$, then it is summable $[C, q, \beta]_k$.

The proof of this theorem follows by an argument similar to that of Theorem 5 of T. M. Flett [4] using the following Lemmas 3, 4 and 5.

Lemma 3. *Let $p>-1$, $\beta>-1$, $p+\beta>-1$, $\delta>0$ and $1\leqslant k<\infty$. If a sequence $\{s_n\}$ is bounded $\{C, p, \beta\}_k$ and summable $\{C, p+1, \beta\}_k$ to the sum O, then it is summable $\{C, p+\delta, \beta\}_k$ to O.*

Lemma 4. (*see A. Zygmund* [9]). *If $p>-1$, $\beta>-1$ and $p+\beta>-1$, then the summability methods (C, p, β) and (C, p) are equivalent.*

Lemma 5. (*see E. Kogbetliantz* [5]). *Let $q>p>-1$. If a sequence $\{s_n\}$ is bounded (C, p) and summable (C) to the sum s, then it is summable (C, q) to the same sum s.*

The implications between the two types { } and [ ] of strong summability methods are established in the following theorems.

3.6. Theorem. 6. *Let $p>-1$, $\beta>-1$, $p+\beta>-1$ and $1\leqslant k\leqslant\infty$. If a sequence $\{s_n\}$ is summable $\{C, p, \beta\}_k$ to the sum s, then it is summable $[C, p+1, \beta]_k$.*

The proof of this theorem follows by an argument similar to that of Theorem 6 of T. M. Flett [4].

3.7. Theorem 7. *Let $p>-1$, $\beta>-1$, $p+\beta>-1$ and $1\leqslant k\leqslant\infty$. If a sequence $\{s_n\}$ is summable $\{C, *, \beta\}_k$ to the sum s (or summable $\{C, *, \beta\}$ to s) and is summable $[C, p+1, \beta]_k$, then it is summable $\{C, p, \beta\}_k$ to s.*

Proof of Theorem 7. Let $1\leqslant k<\infty$. Without loss of generality we may assume that the sum $s=0$. from Theorem 2 (*i*) we get that if a sequence $\{s_n\}$ is summable $\{C, *, \beta\}_k$ to the sum O, then it is summable $\{C, p+\gamma, \beta\}_k$ for some integer γ to O. And from Theorem 2 (iii), we get that if a sequence $\{s_n\}$ is summable $[C, p, \beta]_k$, then it is summable $[C, p+\gamma, \beta]_k$. Hence by the repeated use of the following Lemma 6, the result follows.

Lemma 6. *Let $p>-1$, $\beta>-1$, $p+\beta>-1$ and $1\leqslant k<\infty$. If a sequence $\{s_n\}$ is summable $\{C, p+1, \beta\}_k$ to the sum O and is summable $[C, p+1, \beta]_k$, then it is summable $\{C, p, \beta\}_k$ to O.*

This can be easily proved using (1.7) and Minkowski's inequality.

Consider the case $k=\infty$. From (1.3) we get for $\gamma\geqslant 2$

$$T_{n,\beta}^{p+\gamma}=\frac{1}{E_{n,\beta}^{p+\gamma}}\sum_{r=1}^{n} E_{n-\beta}^{\gamma'-1}\, E_{r,\beta}^{p+1}\, T_{r,\beta}^{p+1} \text{ where } \gamma'=\gamma-1. \tag{3.7.1}$$

From (3.7.1) we get that

$$T_{n,\beta}^{p+1}=o(1), \text{ as } n\to\infty \text{ implies } T_{n,\beta}^{p+\gamma}=o(1), \text{ as } n\to\infty. \tag{3.7.2}$$

From (1.7) we have $C_{n,\beta}^{p+\gamma-1}=(p+\beta+\gamma)^{-1}\,T_{n,\beta}^{p+\gamma}+C_{n,\beta}^{p+\gamma}$.

Hence whenever $T_{n,\beta}^{p+\gamma}\to 0$ and $C_{n,\beta}^{p+\gamma}\to s$, as $n\to\infty$,

(3.7.3) $$C_{n,\beta}^{p+\gamma-1}\to s, \text{ as } n\to\infty.$$

Therefore when $k=\infty$, the result follows from (3.7.2) and the repeated use of (3.7.3). This completes the proof of Theorem 7.

Remark. The conditions in Theorem 7 are also necessary. This follows from Theorems 2 (*i*) and 6.

Theorem 7 is a Tauberian theorem of '*o*' depth. The corresponding Tauberian theorem of '*O*' depth is the following theorem.

3.8. Theorem 8. *Let* $q>p>-1$, $\beta>-1$, $p+\beta>-1$ *and* $1\leqslant k\leqslant\infty$. *If a sequence* $\{s_n\}$ *is summable* $\{C, *, \beta\}_k$ *to the sum s (or summable* $(C, *, \beta)$ *to s) and is bounded* $[C, p+1, \beta]_k$, *then it is summable* $\{C, q, \beta\}_k$ *to s.*

Proof of Theorem 8. If a sequence $\{s_n\}$ is summable $\{C, *, \beta\}_k$, then by Theorem 6 it is summable $[C, *, \beta]_k$. Now since it is also bounded $[C, p+1, \beta]_k$, by Theorem 5 (ii) it is summable $[C, q, \beta]_k$ for every $q>p^+ 1$. From this and Theorem 7 the result follows.

Remark. As a consequence of Theorem 7 and 3 (i) we get that if a sequence $\{s_n\}$ is summable $\{C, *, \beta\}_k$ and summable $[C, p, \beta]_k$, then it is summable (C, p). Further as a consequence of Theorem 8 and Theorem 3 (i) we get that if a sequence $\{s_n\}$ is summable $\{C, *, \beta\}_k$ and bounded $[C, p, \beta]_k$ for $k>1$, then it is summable (C, p), and as a consequence of Theorems 8 and 2 (*i*) we get that if a sequence $\{s_n\}$ is summable $\{C, *, \beta\}_1$ and bounded $[C, p, \beta]_1$, then it is summable $\{C, p, \beta\}_m$ for every finite $m\geqslant 1$.

The case $\beta=0$ of this theorem is a known result of T. M. Flett [4]. The case $\beta=0$ and $k=\infty$ of Theorems 6,7 and 8 are well known results in the theory of ordinary Cesàro summability ([5] pp. 15, 30 and 31).

Now we shall investigate the corresponding results for the generalized Abel — (C, p, β) method $(A_\alpha; C, p, \beta)$.

3.9. Theorem 9. *Let* $p>-1$, $\alpha>-1$, $\beta>-1$ *and* $p+\beta>-1$.

(*i*) *If a sequence* $\{s_n\}$ *is summable* $\{A_\alpha; C, p, \beta\}_k$ *to the sum s and is also summable* $[A_\alpha; C, p, \beta]_k$, *where* $k\geqslant 1$, *then it is summable* $(A_\alpha; C, p, \beta)$ *to the same sum s.*

(*ii*) *If a sequence* $\{s_n\}$ *is summable* $\{A_\alpha; C, p, \beta\}_k$ *to the sum s and is also bounded* $[A_\alpha; C, p, \beta]_k$, *where* $k>1$, *then it is summable* $(A_\alpha; C, p, \beta)$ *to the same sum s.*

(*iii*) *If a sequence* $\{s_n\}$ *is summable* $\{A_\alpha; C, p, \beta\}_1$ *to the sum s and is also bounded* $[A_\alpha; C, p, \beta]_1$, *then it is summable* $\{A_\alpha; C, p, \beta\}_m$ *to the same sum s for every finite* $m\geqslant 1$.

Proof of Theorem 9. Since $A_\alpha^{(p,\beta)}(x)$ is a power series, $A_\alpha^{(p,\beta)}(x)-s$ vanishes only at a finite number of points in $0\leqslant x\leqslant R<1$, so that $\left|A_\alpha^{(p,\beta)}(x)-s\right|$

is differentiable in $(0, R)$ except at a finite number of points. Hence we have for any $k \geqslant 1$, by integration by parts,

$$(3.9.1) \qquad \int_0^R |A_\alpha^{(p,\beta)}(x)-s|^k (1-x)^{-2}\,dx = \left[|A_\alpha^{(p,\beta)}(x)-s|^k (1-x)^{-1}\right]_0^R$$

$$-k\int_0^R |A_\alpha^{(p,\beta)}(x)-s|^{k-1}(1-x)^{-1}\frac{d}{dx}|A_\alpha^{(p,\beta)}(x)-s|\,dx.$$

Since

$$\left|\frac{d}{dx}|A_\alpha^{(p,\beta)}(x)-s|\right| \leqslant \left|\frac{d}{dx}(A_\alpha^{(p,\beta)}(x)-s)\right| = |A_\alpha^{(p,\beta)\prime}(x)|$$

whenever the left side exists, (3.9.1) gives

$$(3.9.2) \quad |A_\alpha^{(p,\beta)}(R)-s|^k \leqslant (1-R)|A_\alpha^{(p,\beta)}(0)-s|^k + (1-R)\int_0^R |A_\alpha^{(p,\beta)}(x)-s|^k (1-$$

$$-x)^{-2}\,dx + k(1-R)\int_0^R |A_\alpha^{(p,\beta)}(x)-s|^{k-1}(1-x)^{-1}|A_\alpha^{(p,\beta)\prime}(x)|\,dx.$$

From (3.9.2) with $k=1$, it follows that if a sequence $\{s_n\}$ is summable $\{A_\alpha; C, p, \beta\}_1$ to the sum s and summable $[A_\alpha; C, p, \beta]_1$, then it is summable $(A_\alpha; C, p, \beta)$ to the sum s and that if it is summable $(A_\alpha; C, p, \beta)_1$ to the sum s and bounded $[A_\alpha; C, p, \beta]_1$, then it is bounded $(A_\alpha; C, p, \beta)$. Now for $1 \leqslant m < < \infty$, we have

$$(3.9.3) \qquad (1-R)\int_0^R |A_\alpha^{(p,\beta)}(x)-s|^m (1-x)^{-2}\,dx \leqslant$$

$$\leqslant \{\sup_{0\leqslant x\leqslant R} |A_\alpha^{(p,\beta)}(x)-s|^{m-1}\}(1-R)\int_0^R |A_\alpha^{(p,\beta)}(x)-s|(1-x)^{-2}\,dx.$$

From (3.9.3) we get that if $\{s_n\}$ is bounded $(A_\alpha; C, p, \beta)$ and summable $\{A_\alpha; C, p, \beta\}_1$ to s, then it is summable $\{A_\alpha; C, p, \beta\}_m$ to s for every finite $m \geqslant 1$. Hence if the sequence $\{s_n\}$ is summable $\{A_\alpha; C, p, \beta\}_1$ and bounded $[A_\alpha; C, p, \beta]_1$ then it is summable $\{A_\alpha, C, p, \beta\}_m$ for every finite $m \geqslant 1$. Therefore the results (i) with $k=1$ and (iii) are proved. Now for $k>1$, we have

$$(3.9.4) \qquad (1-R)\int_0^R |A_\alpha^{(p,\beta)}(x)-s|^{k-1}|A_\alpha^{(p,\beta)\prime}(x)|(1-x)^{-1}\,dx \leqslant$$

$$\leqslant \left\{(1-R)\int_0^R |A_\alpha^{(p,\beta)}(x)-s|^k (1-x)^{-2}\,dx\right\}^{1/k'}$$

$$\cdot\left\{(1-R)\int_0^R (1-x)^{k-2}|A_\alpha^{(p,\beta)\prime}(x)|^k\right\}^{1/k}$$

by Hölder's inequality. If a sequence $\{s_n\}$ is summable $\{A_\alpha; C, p, \beta\}_k$ to s and bounded $[A_\alpha; C, p, \beta]_k$ (or summable $[A_\alpha; C, p, \beta]_k$) for $k>1$, we obtain from (3.9.4) that

$$(3.9.5)\qquad (1-R)\int_0^R \left|A_\alpha^{(p,\beta)}(x)-s\right|^{k-1}(1-x)^{-1}\left|A_\alpha^{(p,\beta)\prime}(x)\right|dx=o(1) \text{ as } R\to 1-0.$$

Hence the results (i) with $k>1$ and (ii) follow from (3 9.5) and (3.9.2) and the proof is completed.

The special case $p=0$ and $\alpha=0$ of this theorem is a known result of T. M. Flett ([4] p. 122, Theorem 9). Theorem 9 (i) includes a known result of B. P. Mishra ([6] p. 313, Theorem 1) as a special case with $\alpha=0$ and $\beta=0$, since by a known result ([6] p. 314, Theorem 4), summability $\{A; C, p\}_k$ and summability $[A; C, p]_k$ are necessary and sufficient for summability $\{A, C; p-1\}_k$. And further Theorem 9 (i) includes a known result of B. P. Mishra ([7] p. 120, Theorem 1) as a special case with $p=0$, since by a known result ([7] p. 120, Theorem 3), summability $\{A_{\alpha+1}\}_k$ implies summability $\{A_\alpha\}_k$, and by another known result ([7] p. 122 Theorem 4), summability $\{A_{\alpha+1}\}_k$ implies summability $[A_\alpha]_k$. Because Theorem 4 of [7] is

"The necessary and sufficient conditions for the sequence $\{s_n\}$ to be summable $\{A_{\alpha+1}\}_k$ to the sum s are that it be summable (A_α) to s and

$$\int_0^Y |yT_\alpha'(y)|^k dy=o(Y), \text{ as } Y\to\infty.\text{"}$$

where

$$T_\alpha(y)=(1+y)^{-(\alpha+1)}\sum_{n=0}^{\infty} E_n^\alpha s_n y^n/(1+y)^n.$$

Hence $yT_\alpha'(y)=x(1-x)A_\alpha'(x)$ where $y=\dfrac{x}{1-x}$, and

$$\int_0^\infty |yT_\alpha'(y)|^k dy=\int_0^1 x^k(1-x)^{k-2}|A_\alpha'(x)|^k dx.$$

Hence $\int_0^R |yT_\alpha'(y)|^k dy=o(Y)$, as $Y\to\infty$, is equivalent to

$$(1-R)\int_0^R x^k(1-x)^{k-2}|A_\alpha'(x)|^k dx=o(1), \text{ as } R\to 1-0,$$

which is equivalent to

$$(1-R)\int_0^R (1-x)^{k-2}|A_\alpha'(x)|^k dx=o(1), \text{ as } R\to 1-0.$$

Hence by definition $\{s_n\}$ is summable $[A_\alpha]_k$.

We shall now pass on to the generalization of the 'o' and 'O' Tauberian theorems for $(A_\alpha; C, p, \beta)$ method.

3.10. Theorem 10. *Let $p>-1$, $q\geq 0$, $\alpha>-1$, $\beta>-1$, $p+\beta>-1$ and $1\leq k\leq\infty$. If a sequence $\{s_n\}$ is summable $\{A_\alpha; C, q, \beta\}_k$ to the sum s and is also summable $[C, p+1, \beta]_k$, then it is summable $\{C, p, \beta\}_k$ to s.*

Proof of Theorem 10. Since $\{s_n\}$ is summable $[C, p+1, \beta]_k$ we obtain by Theorem 4 (ii) that, it summable $[A_\alpha; C, q, \beta]_k$. By hypothesis it is also summable $\{A_\alpha; C, q, \beta\}_k$ to the sum s. Hence we get by Theorem 9 (i) that, it is summable $(A_\alpha; C, q, \beta)$ to the same sum s.

Since $\{s_n\}$ is summable $[C, p+1, \beta]_k$, from Theorem 3 (iii) we get

$$T^{p'}_{n,\beta} = n\left(C^{p'}_{n,\beta} - C^{p'}_{n-1,\beta}\right) = o(1), \text{ as } n \to \infty, \text{ for } p' > p+1, \text{ ie.}$$

$$\left(C^{p'}_{n,\beta} - C^{p'}_{n-1,\beta}\right) = o(n^{-1}), \text{ as } n \to \infty, \text{ for } p' > p+1. \tag{3.10.1}$$

Now summability $(A_\alpha; C, q, \beta)$ of $\{s_n\}$ to the sum s and (3.10.1) imply that it is summable $(C, *, \beta)$ to the same sum s. This follows from the following Lemma 7 which is proved by the author in [10] since the condition (3.10.1) implies the condition (3.10.2) of Lemma 7.

Lemma 7. *Let* $p' > -1$, $q > -1$, $\alpha > -1$, $\beta > -1$, $p' + \beta > -1$ *and* $q + \beta > -1$. *If a real sequence* $\{s_n\}$ *is summable* $(A_\alpha; C, q, \beta)$ *to the sum s and*

$$\underline{\lim} \left(C^{p'}_{n,\beta} - C^{p'}_{m,\beta}\right) \geq 0 \tag{3.10.2}$$

when $n > m$, $m \to \infty$ *so that* $n/m \to 1$, *then* $\{s_n\}$ *is summable* (C, p', β) *to the same sum s.*

Hence by Theorem 7 we obtain that the sequence $\{s_n\}$ is summable $\{C, p, \beta\}_k$ to the sum s. Thus the theorem is established.

Remark. This theorem is stronger than Theorem 7 which is used in the proof. The conditions of this theorem for summability $\{C, p, \beta\}_k$ are also necessary. This part follows from Theorems 4 (i) and 6.

The special case $q=0$, $\alpha=0$ and $\beta=0$ of this theorem is a known result of T. M. Flett ([4] p. 122, Theorem 10). The special case $k=\infty$ of this theorem is the following result in ordinary summability.

3.11. Theorem 11. *Let* $p > -1$, $q > -1$, $\alpha > -1$, $\beta > -1$, $p + \beta > -1$ *and* $q + \beta > -1$. *If a sequence* $\{s_n\}$ *is summable* $(A_\alpha; C, q, \beta)$ *to the sum s and* $\left(C^{p+1}_{n,\beta} - C^{p+1}_{n-1,\beta}\right) = o(n^{-1})$ *as* $n \to \infty$, *then* $\{s_n\}$ *is summable* (C, p, β) *to the same sum s and hence it is summable* (C, p) *to s.*

In Theorem 11, we have $q > -1$ instead of $q \geq 0$, since by Lemma 2 summability $(A_\alpha; C, q, \beta)$ $(q > -1, \beta > -1$ and $q + \beta > -1)$ implies summability $(A_\alpha; C, q', \beta)$ $(q' > q)$, as $\{C^{q'}_{n,\beta}\}$ is a regular Hausdorff transform of $\{C^{q}_{n,\beta}\}$. The last part of Theorem 11 follows from Lemma 4.

Remark. When $q \geq 0$, the conditions for summability (C, p) are also necessary. This follows from Theorems 4 (i) and 6 with $k = m = \infty$.

We have also the following Tauberian theorem which is an immediate consequence of Theorems 10 and 3 (i).

3.12. Theorem 12. *Let* $p>-1$, $q\geqslant 0$, $\alpha>-1$, $\beta>-1$, $p+\beta>-1$ *and either* $q'>p+\frac{1}{k}$ *and* $k>1$ *or* $q'\geqslant p+1$ *and* $k=1$. *If a sequence* $\{s_n\}$ *is summable* $\{A_\alpha;\ C,\ q,\ \beta\}_k$ *to the sum s and is also summable* $[C, p+1, \beta]_k$, *then it is summable* (C, q', β) *to the same sum s and hence it is summable* (C, q') *to s.*

Theorem 10 can be deduced from the following '*O*' Tauberian Theorem 13, but it is more elementary than Theorem 13.

3.13. Theorem 13. *Let* $p'>p>-1$, $q\geqslant 0$, $\alpha>-1$, $\beta>-1$, $p+\beta>-1$ *and* $1\leqslant k\leqslant\infty$. *If a sequence* $\{s_n\}$ *is summable* $\{A_\alpha;\ C,\ q,\ \beta\}_k$ *to the sum s and is either bounded* $\{C, p', \beta\}_k$ *or bounded* $[C, p+1, \beta]_k$, *then it is summable* $\{C', p', \beta\}_k$ *to the same sum s.*

Proof of Theorem 13. By the analogue of Theorem 6 for boundedness, we obtain that, if a sequence $\{s_n\}$ is bounded $\{C, p, \beta\}_k$, then it is bounded $[C, p+1, \beta]_k$. Hence it is bounded $[A_\alpha;\ C,\ q,\ \beta]_k$ by the analogue of Theorem 4 (ii) for boundedness. By hypothesis it is summable $\{A_\alpha;\ C,\ q,\ \beta\}_k$ to s. Hence by Theorem 9 (ii), for $k>1$, we get that, it is summable $(A_\alpha;\ C,\ q,\ \beta)$ to s.

Now boundedness $[C, p+1, \beta]_k$ implies by Theorem 3 (iv) for $p'>p+1$

$$(3.13.1)\qquad \left(C^{p'}_{n,\beta}-C^{p'}_{n-1,\beta}\right)=O(n^{-1}),\ \text{as}\ n\to\infty.$$

Hence boundedness $[C, p+1, \beta]_k$ and summability $(A_\alpha;\ C,\ q,\ \beta)$ of $\{s_n\}$ to the sum s imply that it is summable $(C, *, \beta)$ to s. This follows from Lemma 7, since the condition (3.13.1) implies the condition (3.10.2). Hence the result of the theorem for the case $k>1$ follows from Theorem 8.

If $k=1$, then by Theorem 9 (iii) we obtain that the sequence $\{s_n\}$ is summable $\{A_\alpha;\ C,\ q,\ \beta\}_m$ to the sum s for every finite $m\geqslant 1$. By the analogue of Theorem 2 (iii), we get that boundedness $[C, p+1, \beta]_1$ implies boundedness $[C, *, \beta]_m$ for every finite $m>1$. Hence it implies boundedness $[A_\alpha;\ C,\ q,\ \beta]_m$ by the analogue of Theorem 4 (ii). Now the result for the case $k=1$ follows from the result for the case $k>1$. Hence the theorem is established.

Remark. Theorem 13 is stronger than Theorem 8 which is used in its proof.

The special case $q=0$, $\alpha=0$ and $\beta=0$ of this theorem is a known result of T. M. Flett ([4] p. 122, Theorem 11). The special case $k=\infty$ of this theorem is the following result in ordinary summability.

3.14. Theorem 14. *Let* $p'>p>-1$, $q>-1$, $\alpha>-1$, $\beta>-1$, $p+\beta>-1$ *and* $q+\beta>-1$. *If a sequence* $\{s_n\}$ *is summable* $(A_\alpha;\ C,\ q,\ \beta)$ *to the sum s and is either bounded* (C, p, β) *or* $\left(C^{p+1}_{n,\beta}-C^{p+1}_{n-1,\beta}\right)=O(n^{-1})$, *as* $n\to\infty$, *the* $\{s_n\}$ *is summable* (C, p', β) *to the same sum s and hence it is summable* (C, p') *to s.*

In Theorem 14, we have $q>-1$ instead of $q\geqslant 0$, since by Lemma 2, for $q'>q>-1$, $\beta>-1$ and $q+\beta>-1$, summability $(A_\alpha;\ C,\ q,\ \beta)$ implies summability $(A_\alpha;\ C,\ q',\ \beta)$, as $\{C^{q'}_{n,\beta}\}$ is a regular Hausdorff transform of $\{C^{q}_{n,\beta}\}$. The last part of Theorem 14 follows from Lemma 4.

We have also the following Tauberian theorem which is an immediate consequence of Theorems 13 and 3 (*i*).

3.15. Theorem 15. *Let* $p'>p>-1$, $q\geqslant 0$, $\alpha>-1$, $\beta>-1$, $p+\beta>-1$ *and either* $q'>p'+\frac{1}{k}$ *and* $k>1$ *or* $q'\geqslant p'+1$ *and* $k=1$. *If a sequence* $\{s_n\}$ *is summable* $\{A_\alpha;\ C, q, \beta\}_k$ *to the sum* s *and is either bounded* $\{C, p, \beta\}_k$ *or bounded* $[C, p+1, \beta]_k$, *then it is summable* (C, q', β) *to the same sum* s *and hence it is summable* (C, q') *to* s.

The inequality form of Theorems 10 and 13 is the following Theorem 16 which can be proved by an argument similar to that of Theorem 10.

3.16 Theorem 16. *Let* $p>-1$, $q\geqslant 0$, $\alpha>-1$, $\beta>-1$, $p+\beta>-1$ *and* $1\leqslant k<\infty$. *Then*

$$\sup_N \left\{(N+1)^{-1}\sum_{n=0}^{N}\left|C^p_{n,\beta}\right|^k\right\}^{1/k} \leqslant D(k,p,\beta)\sup_N\left\{(N+1)^{-1}\sum_{n=1}^{N}\left|T^{p+1}_{n,\beta}\right|^k\right\}^{1/k}$$

$$+D(k,q,\alpha,\beta)\sup_R\left\{(1-R)\int_0^R\left|A^{(q,\beta)}_\alpha(x)\right|^k(1-x)^{-2}\,dx\right\}^{1/k}.$$

The special case $q=0$, $\alpha=0$ and $\beta=0$ of this theorem is a known result of T. M. Flett ([3] p. 73, Theorem 5 and [4] p. 122, Theorem 12).

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