THEOREMS ON STRONG SUMMABILITY

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1. Introduction. A sequence \( \{s_n\} \) is said to be summable \((C, p, \beta)\) for \( p > -1, \beta > -1 \) and \( p + \beta > -1 \), to the sum \( s \), if

\[
C^p_{n, \beta} = S^p_{n, \beta}/E^p_{n, \beta} \to s, \text{ as } n \to \infty,
\]

(1.1)

where \( E^p_{n, \beta} \) and \( S^p_{n, \beta} \) are defined by

\[
\sum_{n=0}^{\infty} E^p_{n, \beta} x^n = (1 - x)^{-(p + \beta + 1)} \quad \text{and} \quad \sum_{n=0}^{\infty} S^p_{n, \beta} x^n = (1 - x)^{-p} \sum_{n=0}^{\infty} E^\beta_n s_n x^n.
\]

Here

\[
E^p_{n, \beta} = \sum_{r=0}^{n} E^p_{r, \beta}, \quad S^p_{n, \beta} = \sum_{r=0}^{n} S^p_{r, \beta}, \quad E^p_{n, \beta} = E^p_{n, \beta} = \binom{n + p + \beta}{n} \frac{n^{p + \beta}}{\Gamma(p + \beta + 1)},
\]

\[
E^p_{n, \beta} = \sum_{r=0}^{n} E^p_{n-r} E^p_{r, \beta}, \quad S^p_{n, \beta} = \sum_{r=0}^{n} E^p_{n-r} S^p_{r, \beta}
\]

and

\[
C^p_{n, \beta} = \frac{1}{E^p_{n, \beta}} \sum_{r=0}^{n} E^p_{n-r} E^p_{r, \beta} C^p_{r, \beta} \quad \text{for } \delta > 0.
\]

When \( p > 0 \), \((C, p, \beta)\) method defined by (1.1) is a regular Nörlund method and \([C^p_{n, \beta}]\) is a regular Hausdorff transform of the sequence \( \{s_n\} \) generated by the sequence \( \left\{ \frac{E^\beta_n}{E^p_{n, \beta}} = \frac{\Gamma(p + \beta + 1)}{\Gamma(p + \beta + 1)} \right\} \). If we put \( \beta = 0 \) in the \((C, p, \beta)\) method, we get the familiar Cesàro method \((C, p)\) of order \( p > -1 \). A. Zygmund [9] proved that the methods \((C, p, \beta)\) and \((C, p)\) are equivalent. Summability \((C, O, \beta)\) means the convergence.

For \( p > -1, \beta > -1 \) and \( p + \beta > -1 \) let \( T^p_{n, \beta} = t^p_{n, \beta}/E^p_{n, \beta} \) and

\[
\sum_{n=1}^{\infty} t^p_{n, \beta} x^n = (1 - x)^{-p} \sum_{n=1}^{\infty} E^p_n n (s_n - s_{n-1}) x^n = \sum_{n=0}^{\infty} E^{p-1}_n x^n \sum_{n=1}^{\infty} t^0_{n, \beta} x^n.
\]

(1.2)
Here
\[ t^0_{n, \beta} = E_n^\beta n (s_n - s_{n-1}), \quad t^\beta_{n, \beta} = \sum_{r=1}^{n} E_{n-r}^{\beta-1} t^0_r, \quad t^{\beta+\delta}_{n, \beta} = \sum_{r=1}^{n} E_{n-r}^{\beta-1} t^\beta_r \]
and
\[ T^{\beta+\delta}_{n, \beta} = \frac{1}{E_n^{\beta+\delta}} \sum_{r=1}^{n} E_{n-r}^{\beta-1} E_r^{\beta} T^\beta_r \quad \text{(for } \delta > 0) \]

Let \( f(x) = \sum_{n=0}^{\infty} E_n^\beta s_n x^n \). Then \( \sum_{n=0}^{\infty} S^\beta_{n, \beta} x^n = (1 - x)^{-p} f(x) \). Hence
\[ \sum_{n=0}^{\infty} (p + \beta) S^\beta_{n, \beta} x^n = (p + \beta) (1 - x)^{-p} f(x) \]
and
\[ \sum_{n=0}^{\infty} S^{p-1}_{n, \beta} x^{p+\beta+n} = x^{p+\beta} (1 - x)^{-(p-1)} f(x). \]

Therefore
\[ \sum_{n=0}^{\infty} (p + \beta + n) S^{p-1}_{n, \beta} x^n = x^{-(p+\beta-1)} \frac{d}{dx} \{ x^{p+\beta} (1 - x)^{-(p-1)} f(x) \}. \]

From (1.4) and (1.5) we get
\[ \sum_{n=0}^{\infty} (p + \beta + n) S^{p-1}_{n, \beta} x^n = \sum_{n=0}^{\infty} (p + \beta) S^\beta_{n, \beta} x^n \]
\[ = - (\beta + 1) (1 - x)^{-p} xf(x) + x (1 - x)^{-(p-1)} f'(x) \]
\[ = (1 - x)^{-p} \sum_{n=1}^{\infty} E_n^\beta n (s_n - s_{n-1}) x^n = \sum_{n=1}^{\infty} t^\beta_{n, \beta} x^n \quad \text{by (1.2).} \]

From (1.6) we get \( t^\beta_{n, \beta} = (p + \beta + n) S^{p-1}_{n, \beta} - (p + \beta) S^\beta_{n, \beta} \) and hence
\[ T^\beta_{n, \beta} = (p + \beta) (C^\beta_{n, \beta} - C^{p-1}_{n, \beta}) = n (C^p_{n, \beta} - C^p_{n-1, \beta}). \]

A sequence \( \{s_n\} \) is said to be summable by the generalized Abel method \((A_\alpha)\), for a real number \( \alpha > -1 \), to the sum \( s \), if \( \sum_{n=0}^{\infty} E_n^\alpha s_n x^n \) is convergent for all \( x \) in \( 0 < x < 1 \) and \( \alpha_2 (x) = (1 - x)^{\alpha+1} \sum_{n=0}^{\infty} E_n^\alpha s_n x^n \to s \), as \( x \to 1-0 \). This is briefly denoted by \( s_n \to s \) \((A_\alpha)\). This method was introduced independently by A. Amir Jakimovski ([1] p. 374) and C. T. Rajagopal ([8] p. 93). The properties of this method were discussed in detail by D. Borwein [2]. In the sequence to function transformation method \((A_\alpha)\) if we put \( \alpha = 0 \), we get the familiar Abel method \((A_0)\) or \((A)\).
A sequence \( \{ s_n \} \) is said to be summable by the generalized Abel \(- (C, p, \beta)\) method \((A_{a}; C, p, \beta)\), for \( p > -1, \alpha > -1, \beta > -1 \) and \( p + \beta > -1 \), to the sum \( s \), if
\[
\sum_{n=0}^{\infty} E_n^{n+1} C_{n, \beta} x^n \text{ is convergent for all } x \text{ in } 0 < x < 1 \text{ and }
\]
(1.8) \[
A_{a}^{(p, \beta)}(x) = (1-x)x^{1+} \sum_{n=0}^{\infty} E_n^{n+1} C_{n, \beta} x^n \to s, \text{ as } x \to 1-0, \text{ i.e. } C_{n, \beta} \to s (A_{a}).
\]

The sequence to function transformation method \((A_{a}; C, p, \beta)\) reduces to

(i) generalized Abel-Cesàro method \((A_{a}; C, p)\) when \( \beta = 0 \),
(ii) Abel \(- (C, p, \beta)\) method \((A; C, p, \beta)\) when \( \alpha = 0 \),
(iii) familiar Abel-Cesàro method \((A; C, p)\) when \( \alpha = 0 \) and \( \beta = 0 \),
(iv) generalized Abel method \((A_{a})\) when \( p = 0 \) and
(v) familiar Abel method \((A)\) when \( p = 0 \) and \( \alpha = 0 \).

In this paper strong summability methods \( \{ C, p, \beta \}_{k} \), \( \{ C, p, \beta \}_{k} \), \( \{ A_{a}; C, p, \beta \}_{k} \) and \( \{ A_{a}; C, p, \beta \}_{k} \) based upon summability methods \( \{ C, p, \beta \} \) and \( \{ A_{a}; C, p, \beta \} \) are defined, and various implications between these strong summability methods and the ordinary summability methods \( \{ C, p, \beta \} \), \( \{ C, p \} \) and \( \{ A_{a}; C, p, \beta \} \) are investigated as generalizations of the corresponding results due to T. M. Flett [4] and B. P. Mishra [6 and 7]. The 'o' depth and 'O' depth Tauberian theorems for summability methods \( \{ C, q, \beta \}_{k} \) and \( \{ A_{a}; C, q, \beta \}_{k} \) with summability and boundedness \( \{ C, p + 1, \beta \}_{k} \) as Tauberian conditions are established as generalizations of the corresponding results due to T. M. Flett [4]. In the latter part the 'o' depth and 'O' depth Tauberian theorems for the ordinary summability method \((A_{a}; C, q, \beta)\) with the Tauberian conditions \( (C_{n, \beta}^{p+1} - C_{n-1, \beta}^{p+1}) = o(n^{-1}) \) and \( (C_{n, \beta}^{p+1} - C_{n-1, \beta}^{p+1}) = O(n^{-1}) \) are deduced.

For any number \( k > 1 \) used as an index, we write \( k' = k/(k-1) \), so that \( k \) and \( k' \) are conjugate indices in the sense of Hölder's inequality. \( 1/k' = 0 \) when \( k = 1 \).

We use \( D(a, b, c, \ldots) \) to denote a positive constant depending only on \( a, b, c, \ldots \), not necessarily the same on any two occurrences, \( D \) by itself will denote a positive absolute constant.

Inequalities of the form \( M < D(a, b, c, \ldots) N \) are to be interpreted as meaning "if the expression \( N \) is finite, then the expression \( M \) is also finite and satisfies the inequality".

2. Strong Summability. A sequence \( \{ s_n \} \) is said to be strongly summable \((C, p + 1, \beta)\) with index \( k \), or summable \( \{ C, p, \beta \}_{k} \) to the sum \( s \) for \( p > -1, \beta > -1, p + \beta > -1 \) and \( k > 1 \), if
(2.1) \[
(N + 1)^{-1} \sum_{n=0}^{N} \left| C_{n, \beta} - s \right|^{k} = o(1), \text{ as } N \to \infty.
\]

If (2.1) is true for some \( p > -1 \), then \( \{ s_n \} \) is said to be summable \( \{ C, *, \beta \}_{k} \) to \( s \). The sequence \( \{ s_n \} \) is said to be bounded \( \{ C, *, \beta \}_{k} \) if
(2.2) \[
(N + 1)^{-1} \sum_{n=0}^{N} \left| C_{n, \beta} \right|^{k} = O(1), \text{ as } N \to \infty.
\]

If (2.2) is true for some \( p > -1 \), then \( \{ s_n \} \) is said to be bounded \( \{ C, *, \beta \}_{k} \).
A sequence \( \{s_n\} \) is said to be strongly summable \((A_\alpha; C, p, \beta)\) with index \(k\), or summable \((A_\alpha; C, p, \beta)_k\), to the sum \(s\) for \(p > -1, \alpha > -1, \beta > -1, p + \beta > -1\) and \(k > 1\), if the series \(\sum_{n=0}^{\infty} E_n^\alpha C_{n, \beta} x^n\) is convergent for all \(x\) in \(0 < x < 1\) and \(A_\alpha^{(p, \beta)}(x)\) defined by (1.8) satisfies the condition

\[
(2.3) \quad (1 - R) \int_0^R |A_\alpha^{(p, \beta)}(x) - s|^k (1-x)^{-2} \, dx = o(1), \quad \text{as} \quad R \to 1 - 0.
\]

The sequence \(\{s_n\}\) is said to be bounded \((A_\alpha; C, p, \beta)_k\) if

\[
(2.4) \quad (1 - R) \int_0^R |A_\alpha^{(p, \beta)}(x)|^k (1-x)^{-2} \, dx = O(1), \quad \text{as} \quad R \to 1 - 0.
\]

Summability \((C, p, \beta)_k\) to the sum \(s\) is equivalent to \(|C_{n, \beta}^{p, s}|^k \to 0\) \((C, 1)\), as \(n \to \infty\) and summability \((A_\alpha; C, p, \beta)_k\) to the sum \(s\) is equivalent to \(|A_\alpha^{(p, \beta)}(x) - s|^k \to 0\) \((C, 1)\), as \(x \to 1 - 0\). This follows by integration by parts.

Let \(\gamma > 1\) be a fixed number. Then condition (2.1) is equivalent to

\[
(2.5) \quad \left\{ \sup_{k} \left( \frac{1}{N^{\gamma - 1}} \sum_{n=N}^{\infty} \left| C_{n, \beta}^{p, s} \right|^k (n+1)^{-\gamma} \right) \right\}^{1/k} = o(1).
\]

As \(k \to \infty\) the expression on the left of (2.5) tends to \(n \geq N |C_{n, \beta}^{p, s}|\), so that the limiting form of (2.5) as \(k \to \infty\) is that \(C_{n, \beta}^{p, s} = o(1)\), as \(n \to \infty\). Thus summability \((C, p, \beta)\) may be regarded as the case \(k = \infty\) of summability \((C, p, \beta)_k\).

It is necessary to transform (2.1) into (2.5) in order to obtain a reasonable definition of summability \((C, p, \beta)_k\) for \(k = \infty\). If we take the \((1/k)^{th}\) power of both sides of (2.1) and make \(k \to \infty\), we obtain formally \(n < N \left| C_{n, \beta}^{p, s} \right| = o(1)\), and this implies that \(C_{n, \beta}^{p, s} = s\) for all \(n\). Boundedness \((C, p, \beta)\) may be regarded as the case \(k = \infty\) of boundedness \((C, p, \beta)_k\).

Similarly summability \((A_\alpha; C, p, \beta)\) and boundedness \((A_\alpha; C, p, \beta)\) may be regarded as the case \(k = \infty\) of summability \((A_\alpha; C, p, \beta)_k\) and boundedness \((A_\alpha; C, p, \beta)_k\) respectively.

We shall now define strong summability methods involving the expression \(T_{n, \beta}^{p, s}\). A sequence \(\{s_n\}\) is said to be summable \([C, p, \beta]_k\), for \(p > -1, \alpha > -1, p + \beta > -1\) and \(k > 1\), if

\[
(2.6) \quad (N+1)^{-1} \sum_{n=1}^{N} \left| T_{n, \beta}^{p, s} \right|^k = o(1), \quad \text{as} \quad N \to \infty.
\]

We may also regard the condition

\[
(2.7) \quad T_{n, \beta}^{p, s} = o(1), \quad \text{as} \quad n \to \infty
\]

as the case \(k = \infty\) of the summability \([C, p, \beta]_k\). The sequence \(\{s_n\}\) is said ot be bounded \([C, p, \beta]_k\) if (2.6) or (2.7) holds with \(o\) replaced by \(O\). If (2.6) or (2.7) is true for some \(p > -1\), then the sequence \(\{s_n\}\) is said to be summable \([C, *, \beta]_k\), and similarly in the case of boundedness.
Corresponding to summability \([C, p, \beta]_k\) we have the generalized Abel-\((C, p, \beta)\) summability \([A_x; C, p, \beta]_k\) defined as follows. A sequence \(\{s_n\}\) is said to be summable \([A_x; C, p, \beta]_k\), for \(p > -1, \alpha > -1, \beta > -1, p + \beta > -1\) and \(k > 1\), if

\[
(1 - R) \int_0^R (1 - x)^{k-2} |A_x^{(p, \beta)}(x)|^k dx = o(1), \text{ as } R \to 1 - 0.
\]

When \(k = \infty\), the condition (2.8) is being replaced by

\[
(1 - x) A_x^{(p, \beta)}(x) = o(1), \text{ as } x \to 1 - 0.
\]

The sequence \(\{s_n\}\) is said to be bounded \([A_x; C, p, \beta]_k\), if (2.8) or (2.9) holds with \(o\) replaced by \(O\).

When \(\beta = 0\), summability methods \(\{C, p, \beta\}_k\) and \([C, p, \beta]_k\) reduce respectively to the summability methods \(\{C, p\}_k\) and \(\{c, p\}_k\) defined by T. M. Flett [4].

When \(p = 0\), summability method \(\{A_x; C, p, \beta\}_k\) reduces to summability method \(\{A_x\}_k\). This definition of summability \(\{A_x\}_k\) is equivalent to the definition given by B. P. Mishra [7], being obtained by obvious changes of variable and parameter. When \(\alpha = 0\) and \(p = 0\) summability method \(\{A_x; C, p, \beta\}_k\) reduces to summability method \(\{A\}_k\) defined by T. M. Flett [4]. When \(\alpha = 0\) and \(\beta = 0\) summability method \(\{A_x; C, p, \beta\}_k\) reduces to summability method \(\{A; C, p\}_k\). This definition of summability \(\{A; C, p\}_k\) is equivalent to the definition given by B. P. Mishra [6], being obtained by obvious changes of variable and parameter.

When \(p = 0\), summability method \(\{A_x; C, p, \beta\}_k\) reduces to summability method \(\{A_x\}_k\). The condition to be satisfied by \(\{s_n\}\) for \(\{A_x\}_k\) summability is equivalent to the condition imposed in the known result of B. P. Mishra ([7] p. 122, Theorem 4), being obtained by obvious changes of variable and parameter. When \(\alpha = 0\) and \(p = 0\) summability method \(\{A_x; C, p, \beta\}_k\) reduces to summability method \(\{A\}_k\) defined by T. M. Flett [4], and denoted by him as \(\{A\}_k\). When \(\alpha = 0\) and \(\beta = 0\) summability method \(\{A_x; C, p, \beta\}_k\) reduces to summability method \(\{A; C, p\}_k\). This definition of summability \(\{A; C, p\}_k\) is equivalent to the definition given by B. P. Mishra [6], being obtained by obvious changes of variable and parameter.

3. Theorems. We shall establish the implications between the summability methods defined above.

3.1. Theorem 1. (i) Let \(p > -1, \beta > -1, p + \beta > -1\) and \(1 < k < \infty\). If a sequence \(\{s_n\}\) is summable \(\{C, p, \beta\}_k\) to the sum \(s\), then it is summable \(\{C, p, \beta\}_m\) to the same sum \(s\) for every \(m\) such that \(1 < m < k\).

(ii) Let \(p > -1, \alpha > -1, \beta > -1, p + \beta > -1\) and \(1 < k < \infty\). If a sequence \(\{s_n\}\) is summable \(\{A_x; C, p, \beta\}_k\) to the sum \(s\), then it is summable \(\{A_x; C, p, \beta\}_m\) to the same sum \(s\) for every \(m\) such that \(1 < m < k\).

(iii) If \(p > -1, \alpha > -1, \beta > -1, p + \beta > -1\) and \(0 < m < k < \infty\), then for any \(s\)

\[
\left( (N + 1)^{-1} \sum_{n=0}^{N} |C_{n, \beta} - s|^m \right)^{1/m} \leq \left( (N + 1)^{-1} \sum_{n=0}^{N} |C_{n, \beta} - s|^k \right)^{1/k}
\]

\[
\leq \sup_{n \leq N} |C_{n, \beta} - s|
\]
and

\[(3.1.2) \quad \left\{ (1 - R) \int_0^R \left| A_\alpha^{(p, \beta)} (x) - s \right|^m (1 - x)^{-2} \, dx \right\}^{1/m} \]

\[\leq \left\{ (1 - R) \int_0^R \left| A_\alpha^{(p, \beta)} (x) - s \right|^k (1 - x)^{-2} \, dx \right\}^{1/k} \]

\[\leq \sup_x A_\alpha^{(p, \beta)} (x) - s \leq \sup_n |C_n^{\beta} s - s|.

(iv) Throughout (i)—(iii) we may replace \( \{C, p, \beta\} \) by \([C, p, \beta]\), \( \{A_\alpha; C, p, \beta\} \) by \([A_\alpha; C, p, \beta]\) (with omission of the sum \( s \)), \( \{C_n^{\beta} s - s\} \) by \( T_n^{\beta} \) and \( A_\alpha^{(p, \beta)} (x) - s \) by \((1 - x) A_\alpha^{(p, \beta)} (x)\).

The first inequalities in (3.1.1) and (3.1.2) follow from Hölder's inequality. The second inequalities in (3.1.2) and (3.1.1) are obvious and the third inequality in (3.1.2) follows from the identity

\[A_\alpha^{(p, \beta)} (x) = (1 - x)^{x+1} \sum_{n=0}^{\infty} E_n^{x} C_n^{\beta} x^n \quad \text{for} \ 0 < x < 1,\]

since \((1 - x)^{x+1} \sum_{n=0}^{\infty} E_n^{x} x^n = 1\). The inequalities (3.1.1) and (3.1.2) are analogues of (i) and (ii) for boundedness \( \{C, p, \beta\}_k \) and \( \{A_\alpha; C, p, \beta\}_k \).

Theorem 1 is a collection of elementary results in the direction of decreasing \( k \). The complicated results in the direction of increasing \( k \) are collected together in the following Theorem.

3.2. Theorem 2. Let \( p > -1, \beta > -1, p + \beta > -1 \) and either \( 1 < k < m < \infty \) and \( q > p + \frac{1}{k} - \frac{1}{m} \) or \( 1 - k < m < \infty \) and \( q > p + \frac{1}{k} - \frac{1}{m} \)

(i) If a sequence \( \{s_n\} \) is summable \( \{C, p, \beta\}_k \) to the sum \( s \), then it is summable \( \{C, q, \beta\}_m \) to the same sum \( s \).

(ii) For any \( s \)

\[(3.2.1) \quad \sup_N \left\{ (N + 1)^{-1} \sum_{n=0}^{N} |C_n^{\beta} s - s|^m \right\}^{1/m} \]

\[\leq D (k, m, p, q, \beta) \sup_N \left\{ (N + 1)^{-1} \sum_{n=0}^{N} |C_n^{\beta} s - s|^k \right\}^{1/k}.

(iii) In (i) and (ii) we may replace \( \{C, p, \beta\} \) by \([C, p, \beta]\) \( \{C_n^{\beta} s - s\} \) by \( T_n^{\beta} \) and \( \{C_n^{\beta} s - s\} \) by \( T_n^{\beta} \).

3.3. Theorem 3. Let \( p > -1, \beta > -1, p + \beta > -1 \) and either \( k > 1 \) and \( q > p + 1/k \) or \( k = 1 \) and \( q > p + 1 \).

(i) If a sequence \( \{s_n\} \) is summable \( \{C, p, \beta\}_k \) to the sum \( s \), then it is summable \( \{C, q, \beta\}_m \) and hence \( \{C, q\} \) to the same sum \( s \).
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(ii) For any \( s \)
\[
\text{Sup}_{n} \left| C_{n, \beta}^{\alpha} - s \right| < D(k, p, q, \beta) \text{Sup}_{n} \left\{ \left( N + 1 \right)^{-1} \sum_{n=0}^{N} \left| C_{n, \beta}^{\alpha} - s \right|^{k} \right\}^{1/k}.
\]

(iii) If a sequence \( \{ s_n \} \) is summable \( [C, p, \beta]_k \), then \( T_{n, \beta}^{k} = o(1) \), as \( n \to \infty \).

(iv) \[
\text{Sup}_{n} \left| T_{n, \beta}^{k} \right| < D(k, p, q, \beta) \text{Sup}_{n} \left\{ \left( N + 1 \right)^{-1} \sum_{n=0}^{N} \left| T_{n, \beta}^{p} \right|^{1/k} \right\}^{1/k}.
\]

Theorems 2 and 3 can be proved by the arguments similar to that of Theorems 2 and 3 of T. M. Flett [4].

Remark. From theorems 1 and 3 we get that a sequence \( \{ s_n \} \) is summable \( \{ C, *, \beta \}_{\infty} \) if and only if it is summable \( \{ C, *, \beta \} \).

3.4. Theorem 4. Let \( p > -1, q > 0, \alpha > -1, \beta > -1, p + \beta > -1 \) and \( 1 < k < \infty \).

(i) If a sequence \( \{ s_n \} \) is summable \( \{ C, p, \beta \}_k \) to the sum \( s \), then it is summable \( \{ A_{\alpha}; C, q, \beta \}_m \) to the same sum \( s \) for every \( m (1 < m < \infty) \).

(ii) If a sequence \( \{ s_n \} \) is summable \( \{ C, p, \beta \}_k \), then it is summable \( \{ A_{\alpha}; C, q, \beta \}_m \) for every \( m (1 < m < \infty) \).

Proof of Theorem 4. If a sequence \( \{ s_n \} \) is summable \( \{ C, p, \beta \}_k \) to the sum \( s \), by Theorem 3 (i), it is summable \( \{ C, p' \} \) where \( p' > p + 1/k \) if \( 1 < k \) and \( p' = p + 1 \) if \( k = 1 \), to the same sum \( s \) and hence it is summable \( \{ A_{\alpha} \} \) to \( s \) by the following Lemma 1.

Lemma 1. (see D. Borwein [2].) If a sequence \( \{ s_n \} \) is summable \( \{ C, p \} \) \( (p > -1) \) to the sum \( s \), then it is summable \( \{ A_{\alpha} \} \) \( (\alpha > -1) \) to the same sum \( s \).

Now by the following Lemma 2 we observe that \( \{ s_n \} \) is summable \( \{ A_{\alpha}, C, q, \beta \} \) to \( s \), since \( \{ C_{n, \beta} \} \) is a regular Hausdorff transform of \( \{ s_n \} \).

Lemma 2. (see A. Amir Jakimovski [1]). Let \( \alpha > -1 \) be a real number. If a sequence \( \{ s_n \} \) is summable \( \{ A_{\alpha} \} \) to the same sum \( s \) and \( \{ h_n \} \) is a regular Hausdorff transform of \( \{ s_n \} \), then \( \{ h_n \} \) is summable \( \{ A_{\alpha} \} \) to the same sum \( s \).

Hence by Theorem 1 (ii) with \( k = \infty \), we get that the sequence \( \{ s_n \} \) is summable \( \{ A_{\alpha}; C, q, \beta \}_m \) to the same sum \( s \) for every \( m \). Hence (i) is proved and (ii) follows from (i) applied to the sequence \( \{ n(s_n - s_{n-1}) \} \).

The special case \( q = 0, \alpha = 0 \) and \( \beta = 0 \) of this theorem is a known result of T. M. Flett ([4] p. 120, Theorem 4). The known result of D. Borwein ([2] p. 320, Theorem 4) which is used to prove this theorem is a special case of this theorem with \( q = 0, \beta = 0 \) and \( k = m = \infty \). This theorem also includes the known result of B. P. Mishra ([6] p. 316, Theorem 6) as a special case with \( \alpha = 0, \beta = 0 \) and \( q = p, k = m \). And further the known result of B. P. Mishra ([7] p. 125, Theorem 5) is a particular case of this theorem with \( \beta = 0 \) and \( q = 0 \).

Theorem 4 establishes the connection between summability \( \{ C, p, \beta \}_k \) with summability \( \{ A_{\alpha}; C, q, \beta \}_m \) and summability \( \{ C, p, \beta \}_k \) with summability \( \{ A_{\alpha}; C, q, \beta \}_m \). We have also the following convexity theorem.
3.5. Theorem 5. Let \( q > p > -1, \beta > -1, p + \beta > -1 \) and \( 1 < k < \infty \).

(i) If a sequence \( \{s_n\} \) is bounded \( \{C, p, \beta\}_k \) and summable \( \{C, *, \beta\}_k \) or\( (C, *, \beta) \) to the sum \( s \), then it is summable \( \{C, q, \beta\}_k \) to \( s \).

(ii) If a sequence \( \{s_n\} \) is bounded \( [C, p, \beta]_k \) and summable \( [C, *, \beta]_k \), then it is summable \( [C, q, \beta]_k \).

The proof of this theorem follows by an argument similar to that of Theorem 5 of T. M. Flett [4] using the following Lemmas 3, 4 and 5.

Lemma 3. Let \( p > -1, \beta > -1, p + \beta > -1, \delta > 0 \) and \( 1 < k < \infty \). If a sequence \( \{s_n\} \) is bounded \( \{C, p, \beta\}_k \) and summable \( \{C, p + 1, \beta\}_k \) to the sum \( O \), then it is summable \( \{C, p + \delta, \beta\}_k \) to \( O \).

Lemma 4. (see A. Zygmund [9]). If \( p > -1, \beta > -1 \) and \( p + \beta > -1 \), then the summability methods \( (C, p, \beta) \) and \( (C, p) \) are equivalent.

Lemma 5. (see E. Kogbetliantz [5]). Let \( q > p > -1 \). If a sequence \( \{s_n\} \) is bounded \( (C, p) \) and summable \( (C) \) to the sum \( s \), then it is summable \( (C, q) \) to the same sum \( s \).

The implications between the two types \{\} and \[\] of strong summability methods are established in the following theorems.

3.6. Theorem 6. Let \( p > -1, \beta > -1, p + \beta > -1 \) and \( 1 < k < \infty \). If a sequence \( \{s_n\} \) is summable \( \{C, p, \beta\}_k \) to the sum \( s \), then it is summable \( [C, p + 1, \beta]_k \).

The proof of this theorem follows by an argument similar to that of Theorem 6 of T. M. Flett [4].

3.7. Theorem 7. Let \( p > -1, \beta > -1, p + \beta > -1 \) and \( 1 < k < \infty \). If a sequence \( \{s_n\} \) is summable \( \{C, *, \beta\}_k \) to the sum \( s \) (or summable \( \{C, *, \beta\}_k \) to \( s \)) and is summable \( [C, p + 1, \beta]_k \), then it is summable \( \{C, p, \beta\}_k \) to \( s \).

Proof of Theorem 7. Let \( 1 < k < \infty \). Without loss of generality we may assume that the sum \( s = 0 \). From Theorem 2(i) we get that if a sequence \( \{s_n\} \) is summable \( \{C, *, \beta\}_k \) to the sum \( O \), then it is summable \( \{C, p + \gamma, \beta\}_k \) for some integer \( \gamma \) to \( O \). And from Theorem 2(ii), we get that if a sequence \( \{s_n\} \) is summable \( [C, p, \beta]_k \), then it is summable \( [C, p + \gamma, \beta]_k \). Hence by the repeated use of the following Lemma 6, the result follows.

Lemma 6. Let \( p > -1, \beta > -1, p + \beta > -1 \) and \( 1 < k < \infty \). If a sequence \( \{s_n\} \) is summable \( \{C, p + 1, \beta\}_k \) to the sum \( O \) and is summable \( \{C, p + 1, \beta\}_k \), then it is summable \( \{C, p, \beta\}_k \) to \( O \).

This can be easily proved using (1.7) and Minkowski’s inequality.

Consider the case \( k = \infty \). From (1.3) we get for \( \gamma > 2 \)

\[
(3.7.1) \quad T_{n, \beta}^{p + \gamma} = \frac{1}{E_{n, \beta}^p} \sum_{r=1}^{n} E_{n-\beta}^{\gamma} E_{r, \beta}^{p+1} T_{r, \beta}^{p+1} \text{ where } \gamma = \gamma - 1.
\]

From (3.7.1) we get that

\[
(3.7.2) \quad T_{n, \beta}^{p+1} = o(1), \text{ as } n \to \infty \text{ implies } T_{n, \beta}^{p+\gamma} = o(1), \text{ as } n \to \infty.
\]
From (1.7) we have \( C^{p+\gamma-1}_{n, \beta} = (\rho + \beta + \gamma)^{-1} T^{p+\gamma}_{n, \beta} + C^{p+\gamma}_{n, \beta} \).

Hence whenever \( T^{p+\gamma}_{n, \beta} \to 0 \) and \( C^{p+\gamma}_{n, \beta} \to s \), as \( n \to \infty \),

\[
(3.7.3) \quad C^{p+\gamma-1}_{n, \beta} \to s, \quad \text{as} \quad n \to \infty.
\]

Therefore when \( k = \infty \), the result follows from (3.7.2) and the repeated use of (3.7.3). This completes the proof of Theorem 7.

Remark. The conditions in Theorem 7 are also necessary. This follows from Theorems 2 (i) and 6.

Theorem 7 is a Tauberian theorem of 'o' depth. The corresponding Tauberian theorem of 'O' depth is the following theorem.

3.8. Theorem 8. Let \( q > p > -1, \beta > -1, p + \beta > -1 \) and \( 1 < k < \infty \).

If a sequence \( \{s_n\} \) is summable \( \{C, *, \beta\}_k \) to the sum \( s \) (or summable \( \{C, *, \beta\} \) to \( s \)) and is bounded \( \{C, p + 1, \beta\}_k \), then it is summable \( \{C, q, \beta\}_k \) to \( s \).

Proof of Theorem 8. If a sequence \( \{s_n\} \) is summable \( \{C, *, \beta\}_k \), then by Theorem 6 it is summable \( \{C, *, \beta\}_k \). Now since it is also bounded \( \{C, p + 1, \beta\}_k \), by Theorem 5 (ii) it is summable \( \{C, q, \beta\}_k \) for every \( q > p + 1 \). From this and Theorem 7 the result follows.

Remark. As a consequence of Theorem 7 and 3 (i) we get that if a sequence \( \{s_n\} \) is summable \( \{C, *, \beta\}_k \) and summable \( \{C, p, \beta\}_k \), then it is summable \( \{C, p\} \). Further as a consequence of Theorem 8 and Theorem 3 (i) we get that if a sequence \( \{s_n\} \) is summable \( \{C, *, \beta\}_k \) and bounded \( \{C, p, \beta\}_k \) for \( k > 1 \), then it is summable \( \{C, p\} \), and as a consequence of Theorems 8 and 2 (i) we get that if a sequence \( \{s_n\} \) is summable \( \{C, *, \beta\}_1 \) and bounded \( \{C, p, \beta\}_1 \), then it is summable \( \{C, p, \beta\}_m \) for every finite \( m > 1 \).

The case \( \beta = 0 \) of this theorem is a known result of T. M. Flett [4]. The case \( \beta = 0 \) and \( k = \infty \) of Theorems 6, 7 and 8 are well known results in the theory of ordinary Cesàro summability ([5] pp. 15, 30 and 31).

Now we shall investigate the corresponding results for the generalized Abel — \( \{C, p, \beta\} \) method \( \{A_\alpha; C, p, \beta\} \).

3.9. Theorem 9. Let \( p > -1, \alpha > -1, \beta > -1 \) and \( p + \beta > -1 \).

(i) If a sequence \( \{s_n\} \) is summable \( \{A_\alpha; C, p, \beta\}_k \) to the sum \( s \) and is also summable \( \{A_\alpha; C, p, \beta\}_k \), where \( k > 1 \), then it is summable \( \{A_\alpha; C, p, \beta\} \) to the same sum \( s \).

(ii) If a sequence \( \{s_n\} \) is summable \( \{A_\alpha; C, p, \beta\}_k \) to the sum \( s \) and is also bounded \( \{A_\alpha; C, p, \beta\}_k \), where \( k > 1 \), then it is summable \( \{A_\alpha; C, p, \beta\} \) to the same sum \( s \).

(iii) If a sequence \( \{s_n\} \) is summable \( \{A_\alpha; C, p, \beta\}_1 \) to the sum \( s \) and is also bounded \( \{A_\alpha; C, p, \beta\}_1 \), then it is summable \( \{A_\alpha; C, p, \beta\}_m \) to the same sum \( s \) for every finite \( m > 1 \).

Proof of Theorem 9. Since \( A^{(p, \beta)}_\alpha(x) \) is a power series, \( A^{(p, \beta)}_\alpha(x) - s \) vanishes only at a finite number of points in \( 0 < x < R < 1 \), so that \( |A^{(p, \beta)}_\alpha(x) - s| \)
is differentiable in \((0, R)\) except at a finite number of points. Hence we have for any \(k \geq 1\), by integration by parts,

\[
\int_0^R \left| A^{(p, \beta)}_\alpha(x) - s \right|^k (1-x)^{-2} \, dx = \left[ \left| A^{(p, \beta)}_\alpha(x) - s \right|^k (1-x)^{-1} \right]_0^R \\
- k \int_0^R \left| A^{(p, \beta)}_\alpha(x) - s \right|^{k-1} (1-x)^{-1} \frac{d}{dx} \left| A^{(p, \beta)}_\alpha(x) - s \right| \, dx.
\]

Since

\[
\left| \frac{d}{dx} \left| A^{(p, \beta)}_\alpha(x) - s \right| \right| \leq \left| \frac{d}{dx} \left( A^{(p, \beta)}_\alpha(x) - s \right) \right| = \left| A^{(p, \beta)'}_\alpha(x) \right|
\]

whenever the left side exists, (3.9.1) gives

\[
\int_0^R \left| A^{(p, \beta)}_\alpha(R) - s \right|^k \leq (1-R) \left| A^{(p, \beta)}_\alpha(0) - s \right|^k + (1-R) \int_0^R \left| A^{(p, \beta)}_\alpha(x) - s \right|^{k-1} (1-x)^{-1} \left| A^{(p, \beta)'}_\alpha(x) \right| \, dx.
\]

From (3.9.2) with \(k = 1\), it follows that if a sequence \(\{s_n\}\) is summable \(\{A_\alpha; C, p, \beta\}_1\) to the sum \(s\) and summable \([A_\alpha; C, p, \beta]\), then it is summable \((A_\alpha; C, p, \beta)\) to the sum \(s\) and that if it is summable \((A_\alpha; C, p, \beta)\) to the sum \(s\) and bounded \([A_\alpha; C, p, \beta]\), then it is bounded \((A_\alpha; C, p, \beta)\). Now for \(1 < m < \infty\), we have

\[
(1-R) \int_0^R \left| A^{(p, \beta)}_\alpha(x) - s \right|^m (1-x)^{-2} \, dx \leq \\
\leq \left\{ \sup_{0 < x < R} \left| A^{(p, \beta)}_\alpha(x) - s \right|^{m-1} \right\} (1-R) \int_0^R \left| A^{(p, \beta)}_\alpha(x) - s \right| (1-x)^{-2} \, dx.
\]

From (3.9.3) we get that if \(\{s_n\}\) is bounded \((A_\alpha; C, p, \beta)\) and summable \(\{A_\alpha; C, p, \beta\}_1\) to \(s\), then it is summable \(\{A_\alpha; C, p, \beta\}_m\) to \(s\) for every finite \(m > 1\). Hence if the sequence \(\{s_n\}\) is summable \(\{A_\alpha; C, p, \beta\}_1\) and bounded \([A_\alpha; C, p, \beta]\), then it is summable \(\{A_\alpha, C, p, \beta\}_m\) for every finite \(m > 1\). Therefore the results (i) with \(k = 1\) and (iii) are proved. Now for \(k > 1\), we have

\[
(1-R) \int_0^R \left| A^{(p, \beta)}_\alpha(x) - s \right|^{k-1} \left| A^{(p, \beta)'}_\alpha(x) \right| (1-x)^{-1} \, dx \leq \\
\leq \left\{ (1-R) \int_0^R \left| A^{(p, \beta)}_\alpha(x) - s \right|^k (1-x)^{-2} \, dx \right\}^{1/k'} \\
\cdot \left\{ (1-R) \int_0^R (1-x)^{k-2} \left| A^{(p, \beta)'}_\alpha(x) \right|^k \, dx \right\}^{1/k}
\]
by Hölder's inequality. If a sequence \( \{s_n\} \) is summable \( \{A_{\alpha}; C, p, \beta\}_k \) to \( s \) and bounded \( \{A_{\alpha}; C, p, \beta\}_k \) (or summable \( \{A_{\alpha}; C, p, \beta\}_k \) for \( k > 1 \)), we obtain from (3.9.4) that

\[
(1 - R) \int_0^R |A^{(p, B)}_{\alpha}(x) - s|^{k-1} (1 - x)^{-1} |A^{(p, B)}_{\alpha}(x)| \, dx = o(1) \quad \text{as} \quad R \to 1 - 0.
\]

Hence the results (i) with \( k > 1 \) and (ii) follow from (3.9.5) and (3.9.2) and the proof is completed.

The special case \( p = 0 \) and \( \alpha = 0 \) of this theorem is a known result of T. M. Flett ([4] p. 122, Theorem 9). Theorem 9 (i) includes a known result of B. P. Mishra ([6] p. 313, Theorem 1) as a special case with \( \alpha = 0 \) and \( \beta = 0 \), since by a known result ([6] p. 314, Theorem 4), summability \( \{A; C, p\}_k \) and summability \( \{A; C, p\}_k \) are necessary and sufficient for summability \( \{A, C; p - 1\}_k \).

And further Theorem 9 (i) includes a known result of B. P. Mishra ([7] p. 120, Theorem 1) as a special case with \( p = 0 \), since by a known result ([7] p. 120, Theorem 3), summability \( \{A_{\alpha+1}\}_k \) implies summability \( \{A_{\alpha}\}_k \), and by another known result ([7] p. 122 Theorem 4), summability \( \{A_{\alpha+1}\}_k \) implies summability \( \{A\}_k \).

Because Theorem 4 of [7] is

"The necessary and sufficient conditions for the sequence \( \{s_n\} \) to be summable \( \{A_{\alpha+1}\}_k \) to the sum \( s \) are that it be summable \( \{A_{\alpha}\}_k \) to \( s \) and

\[
\int_y^Y |y T_{\alpha}'(y)|^k \, dy = o(Y), \quad \text{as} \quad Y \to \infty.
\]

where

\[
T_{\alpha}(y) = (1 + y)^{-(\alpha + 1)} \sum_{n=0}^{\infty} E_n \frac{s_n y^n}{(1 + y)^n}.
\]

Hence \( y T_{\alpha}'(y) = x (1 - x) A_{\alpha}'(x) \) where \( y = \frac{x}{1 - x} \), and

\[
\int_0^1 |y T_{\alpha}'(y)|^k \, dy = \int_0^1 x^k (1 - x)^{k-2} |A_{\alpha}'(x)|^k \, dx.
\]

Hence

\[
\int_0^R |y T_{\alpha}'(y)|^k \, dy = o(Y), \quad \text{as} \quad Y \to \infty,
\]

is equivalent to

\[
(1 - R) \int_0^R x^k (1 - x)^{k-2} |A_{\alpha}'(x)|^k \, dx = o(1), \quad \text{as} \quad R \to 1 - 0,
\]

which is equivalent to

\[
(1 - R) \int_0^R (1 - x)^{k-2} |A_{\alpha}'(x)|^k \, dx = o(1), \quad \text{as} \quad R \to 1 - 0.
\]

Hence by definition \( \{s_n\} \) is summable \( \{A\}_k \).

We shall now pass on to the generalization of the 'o' and 'O' Tauberian theorems for \( \{A_{\alpha}; C, p, \beta\}_k \) method.

3.10. Theorem 10. Let \( p > -1 \), \( q > 0 \), \( \alpha > -1 \), \( \beta > -1 \), \( p + \beta > -1 \) and \( 1 < k < \infty \). If a sequence \( \{s_n\} \) is summable \( \{A_{\alpha}; C, q, \beta\}_k \) to the sum \( s \) and is also summable \( \{C, p+1, \beta\}_k \) to \( s \), then it is summable \( \{C, p, \beta\}_k \) to \( s \).
Proof of Theorem 10. Since \( \{s_n\} \) is summable \([C, p + 1, \beta]_k\) we obtain by Theorem 4 (ii) that, it summable \([A_{\alpha}; C, q, \beta]_k\). By hypothesis it is also summable \([A_{\alpha}; C, q, \beta]_k\) to the sum s. Hence we get by Theorem 9 (i) that, it is summable \([A_{\alpha}; C, \alpha, \beta]\) to the same sum s.

Since \( \{s_n\} \) is summable \([C, p + 1, \beta]_k\), from Theorem 3 (iii) we get

\[
T_{n, \beta}^{p_1} = n \left( C_{n, \beta}^{p_1} - C_{n-1, \beta}^{p_1} \right) = o(1), \quad \text{as } n \to \infty, \quad \text{for } p' > p + 1, \text{ i.e.}
\]

(3.10.1) \[
(C_{n, \beta}^{p_1} - C_{n-1, \beta}^{p_1}) = o(n^{-1}), \quad \text{as } n \to \infty, \quad \text{for } p' > p + 1.
\]

Now summability \((A_{\alpha}; C, q, \beta)\) of \( \{s_n\} \) to the sum s and (3.10.1) imply that it is summable \((C, \alpha, \beta)\) to the same sum s. This follows from the following Lemma 7 which is proved by the author in [10] since the condition (3.10.1) implies the condition (3.10.2) of Lemma 7.

Lemma 7. Let \( p' > -1, \quad q > -1, \quad \alpha > -1, \quad \beta > -1, \quad p' + \beta > -1 \) and \( q + \beta > -1 \). If a real sequence \( \{s_n\} \) is summable \((A_{\alpha}; C, q, \beta)\) to the sum s and

(3.10.2) \[
\lim_{n \to \infty} \left( C_{n, \beta}^{p_1} - C_{m, \beta}^{p_1} \right) > 0
\]

when \( n > m, \quad m \to \infty \) so that \( n/m \to 1 \), then \( \{s_n\} \) is summable \((C, p', \beta)\) to the same sum s.

Hence by Theorem 7 we obtain that the sequence \( \{s_n\} \) is summable \([C, p, \beta]_k\) to the sum s. Thus the theorem is established.

Remark. This theorem is stronger than Theorem 7 which is used in the proof. The conditions of this theorem for summability \([C, p, \beta]_k\) are also necessary. This part follows from Theorems 4 (i) and 6.

The special case \( q = 0, \alpha = 0 \) and \( \beta = 0 \) of this theorem is a known result of T. M. Flett ([4] p. 122, Theorem 10). The special case \( k = \infty \) of this theorem is the following result in ordinary summability.

3.11. Theorem 11. Let \( p > -1, \quad q > -1, \quad \alpha > -1, \quad \beta > -1, \quad p + \beta > -1 \) and \( q + \beta > -1 \). If a sequence \( \{s_n\} \) is summable \((A_{\alpha}; C, q, \beta)\) to the sum s and \( (C_{n+1, \beta}^{p_1} - C_{n, \beta}^{p_1}) = o(n^{-1}) \) as \( n \to \infty \), then \( \{s_n\} \) is summable \((C, p, \beta)\) to the same sum s and hence it is summable \((C, p)\) to s.

In Theorem 11, we have \( q > -1 \) instead of \( q > 0 \), since by Lemma 2 summability \((A_{\alpha}; C, q, \beta) (q > -1, \beta > -1 \) and \( q + \beta > -1 \)) implies summability \((A_{\alpha}; C, q', \beta) (q' > q)\), as \( \{C_{n, \beta}^{p_1}\} \) is a regular Hausdorff transform of \( \{C_{n, \beta}^q\} \). The last part of Theorem 11 follows from Lemma 4.

Remark. When \( q > 0 \), the conditions for summability \((C, p)\) are also necessary. This follows from Theorems 4 (i) and 6 with \( k = m = \infty \).

We have also the following Tauberian theorem which is an immediate consequence of Theorems 10 and 3 (i).
3.12. Theorem 12. Let \( p > -1, \ q > 0, \ \alpha > -1, \ \beta > -1, \ p + \beta > -1 \) and either \( q' > p + 1 \) and \( k > 1 \) or \( q' = p + 1 \) and \( k = 1 \). If a sequence \( \{ s_n \} \) is summable \( \{ A ; C, q, \beta \}_k \) to the sum \( s \) and is also summable \( \{ C, p + 1, \beta \}_k \), then it is summable \( \{ C, q', \beta \}_k \) to the same sum \( s \) and hence it is summable \( \{ C, q' \}_s \) to \( s \).

Theorem 10 can be deduced from the following 'O' Tauberian Theorem 13, but it is more elementary than Theorem 13.

3.13. Theorem 13. Let \( p' > p > -1, \ q > 0, \ \alpha > -1, \ \beta > -1, \ p + \beta > -1 \) and \( 1 < k < \infty \). If a sequence \( \{ s_n \} \) is summable \( \{ A ; C, q, \beta \}_k \) to the sum \( s \) and is either bounded \( \{ C, p', \beta \}_k \) or bounded \( \{ C, p + 1, \beta \}_k \), then it is summable \( \{ C', p', \beta \}_k \) to the same sum \( s \).

Proof of Theorem 13. By the analogue of Theorem 6 for boundedness, we obtain that, if a sequence \( \{ s_n \} \) is bounded \( \{ C, p, \beta \}_k \), then it is bounded \( \{ C, p + 1, \beta \}_k \). Hence it is bounded \( \{ A ; C, q, \beta \}_k \) by the analogue of Theorem 4(ii) for boundedness. By hypothesis it is summable \( \{ A ; C, q, \beta \}_k \) to \( s \). Hence by Theorem 9(ii), for \( k > 1 \), we get that, it is summable \( \{ A ; C, q, \beta \}_k \) to \( s \).

Now boundedness \( \{ C, p + 1, \beta \}_k \) implies by Theorem 3(iv) for \( p' > p + 1 \)

\[
(C_{n-1}^{p'} - C_n^{p'}) = O(n^{-1}), \text{ as } n \to \infty.
\]

Hence boundedness \( \{ C, p + 1, \beta \}_k \) and summability \( \{ A ; C, q, \beta \} \) of \( \{ s_n \} \) to the sum \( s \) imply that it is summable \( \{ C, *, \beta \} \) to \( s \). This follows from Lemma 7, since the condition \( (3.13.1) \) implies the condition \( (3.10.2) \). Hence the result of the theorem for the case \( k > 1 \) follows from Theorem 8.

If \( k = 1 \), then by Theorem 9(iii) we obtain that the sequence \( \{ s_n \} \) is summable \( \{ A ; C, q, \beta \}_m \) to the sum \( s \) for every finite \( m > 1 \). By the analogue of Theorem 2(iii), we get that boundedness \( \{ C, p + 1, \beta \}_m \) implies boundedness \( \{ C, *, \beta \}_m \) for every finite \( m > 1 \). Hence it implies boundedness \( \{ A ; C, q, \beta \}_m \) by the analogue of Theorem 4(ii). Now the result for the case \( k = 1 \) follows from the result for the case \( k > 1 \). Hence the theorem is established.

Remark. Theorem 13 is stronger than Theorem 8 which is used in its proof.

The special case \( q = 0, \ \alpha = 0 \) and \( \beta = 0 \) of this theorem is a known result of T. M. Flett ([4] p. 122, Theorem 11). The special case \( k = \infty \) of this theorem is the following result in ordinary summability.

3.14. Theorem 14. Let \( p' > p > -1, \ q > -1, \ \alpha > -1, \ \beta > -1, \ p + \beta > -1 \) and \( q + \beta > -1 \). If a sequence \( \{ s_n \} \) is summable \( \{ A ; C, q, \beta \} \) to the sum \( s \) and is either bounded \( \{ C, p, \beta \} \) or \( (C_{n-1}^{p+1} - C_n^{p+1}, \beta) = O(n^{-1}) \), as \( n \to \infty \), the \( \{ s_n \} \) is summable \( \{ C, p', \beta \} \) to the same sum \( s \) and hence it is summable \( \{ C, p' \} \) to \( s \).

In Theorem 14, we have \( q > -1 \) instead of \( q > 0 \), since by Lemma 2, for \( q' > q > -1, \ \beta > -1 \) and \( q + \beta > -1 \), summability \( \{ A ; C, q, \beta \} \) implies summability \( \{ A ; C, q', \beta \} \), as \( \{ C_{n}^{q'} ; \beta \} \) is a regular Hausdorff transform of \( \{ C_{n}^{q} ; \beta \} \). The last part of Theorem 14 follows from Lemma 4.

We have also the following Tauberian theorem which is an immediate consequence of Theorems 13 and 3(i).
3.15. **Theorem 15.** Let \( p' > p > 1, \ q > 0, \ \alpha > -1, \ \beta > -1, \ p + \beta > -1 \) and either \( q' > p' + \frac{1}{k} \) and \( k > 1 \) or \( q' > p' + 1 \) and \( k = 1 \). If a sequence \( \{s_n\} \) is summable \( \{A_x; C, q, \beta\}_k \) to the sum \( s \) and is either bounded \( \{C, p, \beta\}_k \) or bounded \( \{C, p+1, \beta\}_k \), then it is summable \( (C, q', \beta) \) to the same sum \( s \) and hence it is summable \( (C, q') \) to \( s \).

The inequality form of Theorems 10 and 13 is the following Theorem 16 which can be proved by an argument similar to that of Theorem 10.

3.16 **Theorem 16.** Let \( p > -1, \ q > 0, \ \alpha > -1, \ \beta > -1, \ p + \beta > -1 \) and \( 1 < k < \infty \). Then

\[
\sup_N \left( (N+1)^{-1} \sum_{n=0}^{N} \left| C_{n, \beta}^p \right|^k \right)^{1/k} < D(k, p, \beta) \sup_N \left( (N+1)^{-1} \sum_{n=1}^{N} \left| T_{n, \beta}^{p+1} \right|^k \right)^{1/k}
\]

\[
+ D(k, q, \alpha, \beta) \sup_R \left( (1 - R) \int_0^R \left| A_x^{(q, \beta)}(x) \right|^k (1-x)^{-2} \, dx \right)^{1/k}.
\]

The special case \( q = 0, \ \alpha = 0 \) and \( \beta = 0 \) of this theorem is a known result of T. M. Flett ([3] p. 73, Theorem 5 and [4] p. 122, Theorem 12).

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