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ON A FOURTH ORDER WRONSKIAN ASSOCIATED WITH CLASSICAL ORTHOGONAL POLYNOMIALS.

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1. Introduction. In a work by Karlin and Szegö [1], the general Wronskian and the general Turán expression for orthogonal polynomials have been studied with the condition that the leading coefficient of the n-th order classicial polynomials $\Phi_n(x)$ is to be $(-1)^n K_n$, $(K_n > o)$. The present author in [2] has explicitly evaluated the following fourth order Wronskian

$$W_4(H_n(x)) = \begin{vmatrix} H_n(x) & H_{n-1}(x) & H_{n-2}(x) & H_{n-3}(x) \\ H'_n(x) & H'_{n-1}(x) & H'_{n-2}(x) & H'_{n-3}(x) \\ H''_n(x) & H''_{n-1}(x) & H''_{n-2}(x) & H''_{n-3}(x) \\ H'''_n(x) & H'''_{n-1}(x) & H'''_{n-2}(x) & H'''_{n-3}(x) \end{vmatrix}$$

involving Hermite polynomials $H_n(x)$ in terms of $H_{n-1}(x)$ and its simple zeros, and then established the positivity of $W_4(H_n(x))$.

The purpose of this paper is to show that the fourth order Wronskian $W(\Phi_n(n))$ involving any system of classical orthogonal polynomials $\{\Phi_n(x)\}$ admits of a similar representation i.e. it can be expressed in terms of $\Phi_{n-1}(x)$ and its simple zeros.

2. We require to mention some preliminary results which may be found in any standard texts, viz. [3] or [4]. Notations are mostly adopted from [3].

The system of orthogonal polynomials $\{\Phi_n(x)\}$ is associated with an interval of orthogonality (α, β) and a weight function $\omega(x)$, such that Rodrigues' formula.

$$\Phi_n(x) = \frac{K_n}{\omega(x)} \frac{d^n}{dx^n} (X^n \cdot w(x))$$

is satisfied where K_n is a quantity independent of x and $X = X(x)$. It is also known that as $\{\Phi_n(x)$ is an orthogonal sequence, degree of $X(x)$ must not exceed 2 so that we may suppose

$$X = X(x) = ax^2 + bx + c. \tag{2.1}$$

Also let

$$g_n = \int_{\alpha}^{\beta} \{\Phi_n(x)\}^2 \, \omega(x)\, dx \neq 0,$$

and h_n = the leading coefficient of $\Phi_n(x)$. Besides we know the following recurrence relations which must hold for the system of orthogonal polynomials $\{\Phi_n(x)\}$,

$$A_n \Phi_{n+1}(x) = (x - B_n)\, \Phi_n(x) - C_n \Phi_{n-1}(x), \tag{2.2}$$

$$X\, \Phi'_{n-1}(x) = [\delta_{n-1} + (n-1)\, ax]\, \Phi_{n-1}(x) + \beta_{n-1} \Phi_{n-2}(x), \tag{2.2'}$$

where

$$A_n = h_n / h_{n+1}, \quad C_n = g_n h_{n-1} / g_{n-1} h_n$$

and

$$\delta_n = (n-1)\, X'(0) - \frac{1}{2} X''(x) \frac{h'_{n-1}}{h_{n-1}},$$

$$\beta_{n-1} = -C_{n-1}\left[h_1 K_1 + \left(n - \frac{3}{2}\right) X''(x)\right] \tag{2.3}$$

$$= -C_{n-1}\,[h_1 K_1 + (2n-3)\, a],$$

h'_{n-1} being the coefficient of x^{n-1} in $\Phi_n(x)$.

Let us denote the zeros of $\Phi_{n-1}(x)$ by x_k, $(k = 1, 2, \ldots, n-1)$; then we know that the simple zeros x_k all lie in the interval of orthogonality (α, β) and that

$$\frac{\Phi'_{n-1}(x)}{\Phi_{n-1}(x)} = \sum_{k=1}^{n-1} \frac{1}{x - x_k}. \tag{2.4}$$

Furthermore, we mention the following formula of Ivanoff,

$$\frac{d^n}{dx^n}\left(\frac{g}{h}\right) = \frac{1}{h^{n+1}} \begin{vmatrix} h & 0 & 0 & \cdots & 0 & g \\ h' & h & 0 & \cdots & 0 & g' \\ h'' & 2h' & h & \cdots & 0 & g'' \\ \cdots & \cdots & & \cdots & \cdots & \\ h^n & \binom{n}{1} h^{n-1} & \binom{n}{2} h^{n-2} & \cdots & \binom{n}{n-1} h' & g^{(n)} \end{vmatrix} \tag{2.5}$$

where $h^{(n)} = \dfrac{d^n}{dx^n} h$ and $h \equiv h(x)$, $g \equiv g(x)$.

Now we shall prove the following theorem:

Theorem: *If* $x_k\,(k = 1, 2, 3, \ldots, n-1)$ *be the zeros of* $\Phi_{n-1}(x)$, *then*

$$W_4(\Phi_n(x)) = \frac{12\{\Phi_{n-1}(x)\}^4}{\beta_{n-1}^2 A_{n-1}^2} \cdot \frac{A_{n-1}}{C_{n-2}} \left[\sum_{k=1}^{n-1} \frac{X(x_k)}{(x-x_k)^2} \times \right. \tag{2.6}$$

$$\left. \times \sum_{k=1}^{n-1} \frac{X(x_k)}{(x-x_k)^4} - \left\{\sum_{k=1}^{n-1} \frac{X(x_k)}{(x-x_k)^3}\right\}^2 \right],$$

$\{\Phi_n(x)\}$ *being a sequence of classical orthogonal polynomials.*

3. Proof of the Theorem: Differentiating (2.2) r-times with respect to x, we get

$$(3.1)\qquad A_n D^r \Phi_{n+1}(x) = (x-B_n) D^r \Phi_n(x) - rD^{r-1}\Phi^n(x) - C_n D^r \Phi_{n-1}(x),$$

where

$$D^r \equiv \frac{d^r}{dx^r}.$$

On using (3.1) and (2.2) for $r=1, 2, 3$ and for n replaced by $n-2$, in the 4th column of $W_4(\Phi_n(x))$, we get

$$W_4(\Phi_n(x)) = \frac{1}{C_{n-2}} \begin{vmatrix} \Phi_n & \Phi_{n-1} & \Phi_{n-2} & 0 \\ \Phi_n' & \Phi_{n-1}' & \Phi_{n-2}' & \Phi_{n-2} \\ \Phi_n'' & \Phi_{n-1}'' & \Phi_{n-2}'' & 2\Phi_{n-2}' \\ \Phi_n''' & \Phi_{n-1}''' & \Phi_{n-2}''' & 3\Phi_{n-2}'' \end{vmatrix}.$$

On making similar operations on the third column, we get

$$W_4(\Phi_n(x)) C_{n-1} C_{n-2} = \begin{vmatrix} \Phi_n & \Phi_{n-1} & 0 & 0 \\ \Phi_n' & \Phi_{n-1}' & \Phi_{n-1} & \Phi_{n-2} \\ \Phi_n'' & \Phi_{n-1}'' & 2\Phi_{n-1}' & 2\Phi_{n-2}' \\ \Phi_n''' & \Phi_{n-1}''' & 3\Phi_{n-1}'' & 3\Phi_{n-2}'' \end{vmatrix}$$

$$W_4(\Phi_n(x)) C_{n-1} C_{n-2} = \frac{2\alpha_{n-1}}{\{\Phi_{n-1}\}^2} \begin{vmatrix} \Phi_n & \Phi_{n-1} & 0 & 0 \\ \Phi_n' & \Phi_{n-1}' & \Phi_{n-1} & 0 \\ \Phi_n'' & \Phi_{n-1}'' & 2\Phi_{n-1}' & \Phi_{n-1} \\ \Phi_n''' & \Phi_{n-1}''' & 3\Phi_{n-1}'' & \frac{3\alpha_{n-1}'}{2\alpha_{n-1}}\Phi_{n-1} \end{vmatrix}.$$

where

$$\alpha_{n-1} = \Phi_{n-1}\Phi_{n-2}' - \Phi_{n-1}'\Phi_{n-2}$$

$$(3.2)\qquad = \{\Phi_{n-1}\}^2 \frac{d}{dx}\left(\frac{\Phi_{n-2}}{\Phi_{n-1}}\right)$$

and

$$\alpha_{n-1}' = \frac{d}{dx}\alpha_{n-1}.$$

Thus we get

$$(3.3)\qquad \frac{W_4(\Phi_n(x)) C_{n-1} C_{n-2}\{\Phi_{n-1}\}^2}{2\alpha_{n-1}} = \begin{vmatrix} \Phi_n & \Phi_{n-1} & 0 & 0 \\ \Phi_n' & \Phi_{n-1}' & \Phi_{n-1} & 0 \\ \Phi_n'' & \Phi_{n-1}'' & 2\Phi_{n-1}' & \Phi_{n-1} \\ \Phi_n''' & \Phi_{n-1}''' & 3\Phi_{n-1}'' & \frac{3\alpha_{n-1}'}{2\alpha_{n-1}}\Phi_{n-1} \end{vmatrix}$$

Now by (2.5), we get

$$(3.4)\qquad \frac{d^3}{dx^3}\left(\frac{\Phi_n}{\Phi_{n-1}}\right)=-\frac{1}{\{\Phi_{n-1}\}^4}\begin{vmatrix}\Phi_n & \Phi_{n-1} & 0 & 0\\ \Phi_n' & \Phi_{n-1}' & \Phi_{n-1} & 0\\ \Phi_n'' & \Phi_{n-1}'' & 2\,\Phi_{n-1}' & \Phi_{n-1}\\ \Phi_n''' & \Phi_{n-1}''' & 3\,\Phi_{n-1}''' & 3\,\Phi_{n-1}'\end{vmatrix}$$

$$(3.5)\qquad \frac{d^2}{dx^2}\left(\frac{\Phi_n}{\Phi_{n-1}}\right)=\frac{1}{\{\Phi_{n-1}\}^3}\begin{vmatrix}\Phi_n & \Phi_{n-1} & 0\\ \Phi_n' & \Phi_{n-1}' & \Phi_{n-1}\\ \Phi_n'' & \Phi_{n-1}'' & 2\,\Phi_{n-1}'\end{vmatrix}$$

$\therefore$ from (3.3), (3.4), (3.5),

$$(3.6)\qquad \begin{aligned}&\frac{W_4(\Phi_n(x))\,C_{n-1}\,C_{n-2}\{\Phi_{n-1}\}^2}{2\,\alpha_{n-1}}+\{\Phi_{n-1}\}^4\frac{d^3}{dx^3}\left(\frac{\Phi_n}{\Phi_{n-1}}\right)\\ &=\{\Phi_{n-1})^3\times\frac{3(\alpha_{n-1}'\,\Phi_{n-1}-2\,\Phi_{n-1}'\,\alpha_{n-1}}{2\,\alpha_{n-1}}\cdot\frac{d^2}{dx^2}\left(\frac{\Phi_n}{\Phi_{n-1}}\right).\end{aligned}$$

But we know from (3.2) that

$$\alpha_{n-1}=\{\Phi_{n-1}\}^2\frac{d}{dx}\left(\frac{\Phi_{n-2}}{\Phi_{n-1}}\right)$$

and

$$(3.7)\qquad \frac{d^2}{dx^2}\left(\frac{\Phi_{n-2}}{\Phi_{n-1}}\right)=\frac{d}{dx}\left(\frac{\alpha_{n-1}}{\{\Phi_{n-1}\}^2}\right)=\frac{\alpha_{n-1}'\,\Phi_{n-1}-2\,\Phi_{n-1}'\,\alpha_{n-1}}{\{\Phi_{n-1}\}^3}.$$

But as from (2.2)

$$A_{n-1}\,\Phi_n=(x-B_{n-1})\,\Phi_{n-1}-C_{n-1}\,\Phi_{n-2},$$

so,

$$(3.8)\qquad A_{n-1}\frac{d}{dx}\left(\frac{\Phi_n}{\Phi_{n-1}}\right)=1-C_{n-1}\frac{d}{dx}\left(\frac{\Phi_{r-2}}{\Phi_{n-1}}\right)\ \text{and}$$

$$A_{n-1}\frac{d}{dx}\left(\frac{\Phi_n}{\Phi_{n-1}}\right)=-C_{n-1}\frac{d^2}{dx^2}\left(\frac{\Phi_{n-2}}{\Phi_{n-1}}\right).$$

From (3.6)

$$W_4(\Phi_n(x)\,C_{n-1}\,C_{n-2}\{\Phi_{n-1}\}^2+2\,\alpha_{n-1}\{\Phi_{n-1}\}^4\frac{d^3}{dx^3}\left(\frac{\Phi_n}{\Phi_{n-1}}\right)=$$

$$=3\{\Phi_{n-1}\}^6\frac{d^2}{dx^2}\left(\frac{\Phi_{n-2}}{\Phi_{n-1}}\right)\cdot\frac{d^2}{dx^2}\left(\frac{\Phi_n}{\Phi_{n-1}}\right)$$

which by virtue of (3.8), (3.7) becomes

$$W_4(\Phi_n(x))\,C_{n-1}\,C_{n-2}\,\{\Phi_{n-1}\}^2 = 3\,\{\Phi_{n-1}\}^6\,\frac{d^2}{dx^2}\left(\frac{\Phi_{n-2}}{\Phi_{n-1}}\right)\cdot\frac{d^2}{dx^2}\left(\frac{\Phi_n}{\Phi_{n-1}}\right)-$$

$$-2\left\{1-A_{n-1}\,\frac{d}{dx}\left(\frac{\Phi_n}{\Phi_{n-1}}\right)\right\}\times\{\Phi_{n-1}\}^6\times\frac{d^3}{dx^3}\left(\frac{\Phi_n}{\Phi_{n-1}}\right)$$

$$\therefore\quad \frac{W_4(\Phi_n(x))\,C_{n-1}\,C_{n-2}}{\{\Phi_{n-1}\}^4} = -2\,\frac{d^3}{dx^3}\left(\frac{\Phi_n}{\Phi_{n-1}}\right)+A_{n-1}\left[2\,\frac{d^3}{dx^3}\left(\frac{\Phi_n}{\Phi_{n-1}}\right)\cdot\frac{d}{dx}\left(\frac{\Phi_n}{\Phi_{n-1}}\right)-\right.$$

$$\left.-2\left\{\frac{d^2}{dx^2}\left(\frac{\Phi_n}{\Phi_{n-1}}\right)\right\}^2\right]. \tag{3.9}$$

From (2.2) and (2.2′), we have the following relation

$$X\,C_{n-1}\,\Phi'_{n-1}(x)+\beta_{n-1}\,A_{n-1}\,\Phi_n(x)=\Phi_{n-1}(x)\,[\pi_{n-1}+\lambda_{n-1}\,x]$$

where

$$\begin{aligned}\lambda_{n-1}&=(n-1)\,a\,C_{n-1}+\beta_{n-1}\\ \pi_{n-1}&=\delta_{n-1}\,C_{n-1}-B_{n-1}\,\beta_{n-1}\end{aligned} \tag{3.10}$$

$$\therefore\quad C_{n-1}\,X\,\frac{\Phi'_{n-1}(x)}{\Phi_{n-1}(x)}+\beta_{n-1}\,A_{n-1}\,\frac{\Phi_n(x)}{\Phi_{n-1}(x)}=\pi_{n-1}+\lambda_{n-1}\cdot x$$

which along with (2.4), gives

$$\beta_{n-1}\,A_{n-1}\,\frac{\Phi_n}{\Phi_{n-1}}=\pi_{n-1}+\lambda_{n-1}\,x-C_{n-1}\,X\sum_{k=1}^{n-1}\frac{1}{x-x_k}$$

$$\therefore\quad \beta_{n-1}\,A_{n-1}\,\frac{d}{dx}\left(\frac{\Phi_n}{\Phi_{n-1}}\right)=\lambda_{n-1}+C_{n-1}\,X\sum_{k=1}^{n-1}\frac{1}{(x-x_k)^2}-C_{n-1}\,X'\sum_{k=1}^{n-1}\frac{1}{(x-x_k)}=$$

$$=\lambda_{n-1}+C_{n-1}\sum_{k=1}^{n-1}\frac{-a\,(x-x_k)^2+ax_k^2+bx_k+c}{(x-x_k)^2}=$$

$$=\lambda_{n-1}-a\,(n-1)\,C_{n-1}+C_{n-1}\sum_{k=1}^{n-1}\frac{X(x_k)}{(x-x_k)^2}=$$

$$=\beta_{n-1}+C_{n-1}\sum_{k=1}^{n-1}\frac{X(x_k)}{(x-x_k)^2},\quad\text{by (3.10).} \tag{3.11}$$

$$\therefore\quad \beta_{n-1}\,A_{n-1}\,\frac{d^2}{dx^2}\left(\frac{\Phi_n}{\Phi_{n-1}}\right)=-2\,C_{n-1}\sum_{k=1}^{n-1}\frac{X(x_k)}{(x-x_k)^3}, \tag{3.11′}$$

$$\beta_{n-1}\,A_{n-1}\,\frac{d^3}{dx^3}\left(\frac{\Phi_n}{\Phi_{n-1}}\right)=-6\,C_{n-1}\sum_{k=1}^{n-1}\frac{X(x_k)}{(x-x_k)^4}. \tag{3.11″}$$

Applying (3.11), (311′) and (3.11″) to (3.9), we have

$$\frac{W_4(\Phi_n(x))\,C_{n-1}^2\,C_{n-2}}{\{\Phi_{n-1}(x)\}^4}=$$

$$=\frac{12\,C_{n-1}^2}{\beta_{n-1}^2\,A_{n-1}}\left[\sum_{k=1}^{n-1}\frac{X(x_k)}{(x-x_k)^2}\cdot\sum_{k=1}^{n-1}\frac{X(x_k)}{(x-x_k)^4}-\left\{\sum_{k=1}^{n-1}\frac{X(x_k)}{(x-x_k)^3}\right\}^2\right]$$

which establishes the Theorem.

Now we proceed to study the special cases

Case 1. Hermite Polynomials $\{H_n(x)\}$:

For this sequence of polynomials,

$$A_{n-1}=\frac{1}{2}, \quad C_{n-2}=n-2,$$

$$\beta_{n-1}=2(n-1), \quad \text{and} \quad X(x)=1.$$

$$W_4(H_n(x))=\frac{6\{H_{n-1}(x)\}^4}{(n-2)(n-1)^2}\left[\sum_{k=1}^{n-1}\frac{1}{(x-x_k)^2}\cdot\sum_{k=1}^{n-1}\frac{1}{(x-x_k)^4}-\left\{\sum_{k=1}^{n-1}\frac{1}{(x-x_k)^3}\right\}^2\right]$$

which is, thus, positive. This result has already been established by the author [2].

Case 2. Laguerre Polynomials $\{L_n^{(\alpha)}(x)\}$:

For this sequence of polynomials,

$$A_{n-1}=-n, \qquad C_{n-2}=-(\alpha+n-1)$$

$$\therefore \quad A_{n-1}/C_{n-2}=n/(\alpha+n-1) \quad \text{is positive.}$$

Now x_k's all lie in the interval of orthogonality $(0, \infty)$ and $X(x)=x$, so $X(x_k)=x_k$ is positive. Hence $W_4(L_n^{(\alpha)}(x))$ is positive for x in $(0, \infty)$.

Case 3. Jacobi Polynomials $\{P_n^{(\alpha,\beta)}(x)\}$:

For this sequence of polynomials A_{n-1} and C_{n-2} are both positive and $X(x)=x^2-1$. As x_k's all lie on the interval of orthogonality $(-1, 1)$, so $X(x_k)=x_k^2-1$ is positive for $k=1, 2, 3, \ldots, (n-1)$. Hence $W_4(P_n^{(\alpha,\beta)}(x))$ is positive for $x\in(-1, 1)$.

As Ultraspherical Polynomials $\{P_n^{(\lambda)}(x)\}$ and Legendre polynomials $P_n(x)$ may be treated as particular cases of $\{P_{n}^{(\alpha,\beta)}(n)\}$, so these two are not considered separately.

Thus for a sequence $\{\Phi_n(x)\}$ of classical orthogonal polynomials, $W_4(\Phi_n(n))$ is positive for x lying in the interval of orthogonality.

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