PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série tome 18 (32), 1975, pp, 57—69

FIXED AND PERIODIC POINTS FOR A CLASS OF CONTRACTIVE OPERATORS

Ljubomir B. Ćirić

(Communicated October 29, 1974)

Introduction. Let (M, d) be a metric space. An operator $A:M\to M$ is a *contraction mapping* iff for every $x, y\in M$

$$x\neq y \text{ implies } d(Ax, Ay)<q\cdot d(x, y) \tag{1}$$

for some $q<1$. If $q=1$, then A is called a *contractive mapping.*

A contractive mapping is clearly uniformly continuous, and if such operator has a fixed point, then this fixed point is obviously unique. But contractivity of A is not sufficient for the existence of a fixed point in a complete metric space. For example, let $M=[0, \infty)\subset R$ and let $A:M\to M$ be defined by $Ax=x+e^{-x}$. Then $d(Ax, Ay)=(1-e^{-c})\, d(x, y)$ for some $c=x+t(y-x)$; $0<t<1$. Clearly, A is contractive without a fixed point.

If M is a compact metric space, then every contractive operator on M has a fixed point. This fact is motive for the false assertion that a contractive mapping on a compact space is always a contraction for some q (see [10], pp 565). That this assertion is incorrect following example shows. Let $M=[0, 1]\subset R$ and let $A:M\to M$ be defined by $Ax=\frac{n}{n+2}x+\frac{1}{(n+1)(n+2)}$ for $x\in\left[\frac{1}{n+1}, \frac{1}{n}\right]$ $(n=1, 2, 3, \ldots)$ and $A(0)=0$. Clearly, A is contractive on the compact space $[0, 1]$. But for every fixed q $(0<q<1)$, $n>\frac{2q}{1-q}$ and $x, y\in\left[\frac{1}{n+1}, \frac{1}{n}\right]$, $x\neq y$, imply $d(Ax, Ay)=\frac{n}{n+2}d(x, y)>q\cdot d(x, y)$.

M. Edelstein [6] has investigated contractive mappings on spaces which are not necerarily compact. Edelstein has proved that the assumption that there exists a point $x\in M$ such that its sequence of iterates contains a subsequence which converges to a point of M is sufficient for the existence of a fixed point

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This paper is based on the second part of the author's unpublished doctoral dissertation at the University of Belgrade, 1970.

of a contractive mapping, and periodic points of an ε-contractive mapping, respectivelly. Sufficient conditions for the existence of a unique fixed point for ε-contractive mappings were given by Edelstein [6], too.

D. Bailey in his doctoral dissertation [1] (see [2]) has investigated mappings satisfying any of the following conditions:

(2) A mapping A is continuous and $0<d(x, y)$ implies

$$\exists\, n(x, y) \in N: d(A^n x, A^n y)<d(x, y).$$

(3) A is continuous and $\exists\, \varepsilon>0: 0<d(x, y)<\varepsilon$ implies

$$\exists\, n(x, y) \in N: d(A^n x, A^n y)<d(x, y).$$

It is obvious that contractive and ε-contractive operators satisfy (2) or (3), respectively, with $m(x, y)=1$. Bailey has shown that on compact spaces mappings satisfying (2) have a unique fixed point and mappings satisfying (3) have periodic points.

In this paper we introduce the notions of the following mappings of contractive type and prove several fixed-point or periodic-point theorems.

(4) A will be called an *eventually conctractive operator* iff for every $x, y \in M$ there exists $m(x, y) \in N$ such that

$$0<d(x, y) \text{ implies } d(A^n x, A^n y)<d(x, y) \text{ for all } n \geqslant m(x, y).$$

If $d(A^n x, A^n y)<d(x, y)$ is replaced by $d(A^n x, A^n y) \leqslant d(x, y)$, then A will be called *eventually non-expànsive.*

(5) A will be called an *eventually ε-conctractive* [resp. *ε-nonexpansive*] *operator* if there exist $\varepsilon>0$ (ε is const.) and $m(x, y) \in N$ such that

$$0<d(x, y)<\varepsilon \text{ implies } d(A^n x, A^n y)<[\text{resp} \leqslant]\, d(x, y) \text{ for all } n \geqslant m(x, y).$$

(6) A will be called a *frequently contractive* [resp. *nonexpansive*] *operator* f for every $x, y \in M$ there exists $m(x, y) \in N$ such that

$$0<d(x, y) \text{ implies } d(A^{m(x,y)} x, A^{m(x,y)} y)<[\text{resp} \leqslant]\, d(x, y).$$

(7) A will be called a *frequently ε-contractive* [resp. *ε-nonexpansive*] *operator* if there exist $\varepsilon>0$ and $m(x, y) \in N$ such that

$$0<d(x, y)<\varepsilon \text{ imply } d(A^{m(x,y)} x, A^{m(x,y)} y<[\text{resp} \leqslant]\, d(x, y).$$

(8) A will be called with a *contractive* [resp. *nonexpansive*] *iteration at a point* if for each $x \in M$ there exists $m(x) \in N$ such that

$$0<d(x, y) \text{ implies } d(A^{m(x)} x, A^{m(x)} y)<[\text{resp} \leqslant]\, d(x, y).$$

(9) A will be called with an *ε-contractive* [resp. *ε-nonexpansive*] *iteration at a point* if there exist $\varepsilon>0$ and $m(x) \in N$ such that

$$0<d(x, y)<\varepsilon \text{ implies } d(A^{m(x)} x, A^{m(x)} y)<[\text{resp} \leqslant]\, d(x, y).$$

It is obvious that a contractive mapping is continuous (uniformly) and satisfies (4) with $m(x, y)=1$. An eventually contractive mapping is not necessarily continuous, as shows the example 1 which follows Theorem 1. Mappings (6) and (7) satisfy only the contractive condition in (2) and (3), respectively, and they are not necessarily continuous. Example 4, following Theorem 6, shows that results concerning mappings (6) and (7), here presented, are indeed extension of the main results of Bailey [1], [2]. Obviously, mappings (8) and (9) satisfy the conditions for mappings (7) and (8), respectively. As these mappings have certain good properties, we point them out.

1. Fixed and periodic points of eventually contractive operators.

1. Now we shall indicate sufficient conditions for the existence of a unique fixed point for eventually contractive operators.

Let $A: M \to M$ be an operator on a metric space (M, d) and let x be in M. Denote

$$0(x, A) = \{x, Ax, A^2 x, \dots\},$$

$$L(x, A) = \{p \in M : p \text{ is a cluster point of } 0(x, A)\},$$

$$L(M, A) = \bigcup_{x \in M} L(x, A).$$

Recall that A is said to be *orbitally continuous* if A continuous on $\overline{0}(x, A) = \mathrm{cl}\,[0(x, A)]$ for every $x \in M$.

Theorem 1. *Let M be a metric space and A an eventually non-expansive operator on M. If $L(x_0, A)$ is nonempty for some $x_0 \in M$ and A is eventually contractive on $L(M, A)$ and orbitally continuous on M, then A has a unique fixed point $p \in M$ and $p = \lim_n A^n x_0$.*

Proof. Let $p \in L(x_0, A)$. Then there exists a subsequence

$$N_1 = \{n_i : i \in N\}\,(n_1 < n_2 < \cdots n_i < \cdots)$$

of N such that

$$p = \lim_i A^{n_i} x_0.$$

By orbitally continuity of A it is easy to prove that

(10) $$A^s p = \lim_i A^s A^{n_i} x_0 = \lim_i A^{n_i + s} x_0 \quad \text{for each } s = 1, 2, 3 \dots$$

Assume that $Ap \neq p$. Since $p, Ap \in L(x_0, A) \subseteq L(M, A)$ and A is eventually contractive on $L(M, A)$, we may choose a positive integer $r \geq m(p, Ap)$ such that

(11) $$d(A^r p, A^r Ap) < d(p, Ap).$$

Consider $A^{n_i + r} x_0$ and $A^{n_i + r + 1} x_0$ for each fixed $n_i \in N_1$. As A is eventually non-expansive on M, there exists an integer $m(A^{n_i + r} x_0, A^{n_i + r + 1} x_0)$ such that

(12) $$d(A^n x_0, A^{n+1} x_0) = d(A^{n - n_i - r} A^{n_i + r} x_0, A^{n - n_i - r} A^{n_i + r + 1} x_0) \leq d(A^{n_i + r} x_0, A^{n_i + r + 1} x_0)$$

for all $n \geq m(A^{n_i + r} x_0, A^{n_i + r + 1} x_0) + n_i + r$.

For each fixed $n_i \in N_1$ let $n_{j_i} \in N_1$ be chosen such that

$$n_{j_i} \geq m(A^{n_i + r} x_0, A^{n_i + r + 1} x_0) + n_i + r.$$

Then by (12)

(13) $$d(A^{n_{j_i}} x_0, A^{n_{j_i} + 1} x_0) \leq d(A^{n_i + r} x_0, A^{n_i + r + 1} x_0).$$

We may assume that $n_{j_i} < n_{j_{i+1}}$ $(i = 1, 2, 3, \ldots)$. Since $\{A^{n_{j_i}} x_0\}$ is a subsequence of $\{A^{n_i} x_0\}$ and $\lim_i A^{n_i} x_0 = p$ it follows that $\lim_i A^{n_{j_i}} x_0 = p$. Thus we obtain (using (10)) that

$$\lim_i d(A^{n_{j_i}} x_0, A^{n_{j_i}+1} x_0\} = d(p, Ap),$$
$$\lim_i d(A^{n_i+r} x_0, A^{n_i+r+1} x_0) = d(A^r p, A^{r+1} p).$$

Then from (13) it follows that

$$d(p, Ap) \leqslant d(A^r p, A^{r+1} p) = d(A^r p, A^r Ap).$$

This contradicts the condition (11). Thus we conclude that $Ap = p$. Assume now that $q = Aq$ and $q \neq p$. Then $p, q \in L(M, A)$ and by (4) there is $m(p, q) \in N$ such that $d(p, q) = d(A^n p, A^n q) < d(p, q)$ for all $n \geqslant m(p, q)$. This contradiction shows that p is a unique fixed point of A. Finally, to show that $\lim_n A^n x_0 = p$, let $\varepsilon > 0$ be an arbitrary real number. Since $p \in L(x_0, A)$, there is $k \in N$ such that $d(p, A^k x_0) < \varepsilon$. By (4)

$$d(p, A^n x_0) = d(A^n p, A^n x_0) \leqslant d(p, A^k p) < \varepsilon$$

for all $n \geqslant m(p, A^k x_0)$. Thus we conclude that $\lim_n A^n x_0 = p$ and the proof of the Theorem 1 is complete.

Corollary 1.1. *If M is a compact metric space and A is a continuous and eventually contractive mapping on M, then A has a unique fixed point $p \in M$ and $\lim_n A^n x = p$ for every $x \in M$.*

Corollary 1.2. (Edelstein [6], Th. 1.) *Let X be a metric space, f a contractive selfmapping of X satisfying*

(14) $$\exists x (\in X) : \{f^n(x)\} \supset \{f^{n_i}(x)\} \text{ with } \lim_i f^{n_i}(x) \in X,$$

then $\xi = \lim_i f^{n_i}(x)$ is a unique fixed point.

Observe that an eventually contractive operator is not necessarilly orbitally continuous and may be without a fixed point, even if M is compact and connected, as is shown by following example:

Example 1. Let $M = [0, 1] \subset R$ and let $A : M \to M$ be defined by

$$A(0) = 1 \text{ and } A(x) = \frac{x}{2}, \text{ if } x \neq 0.$$

Then $d(Ax, Ay) \leqslant \frac{1}{2} \cdot d(x, y)$ if $x \cdot y \neq 0$ and $d(A^n x, A^n 0) < d(x, 0)$ for all $n \geqslant m(x, 0) = E\left(\log_2 \frac{4}{x}\right)$. Thus A is eventually contractive on M. But A has no fixed point and A is not orbitally continuous on M, because for

$$x \neq 0 : 0 = \lim_n A^n x \text{ and } A(0) = 1 \neq \lim_n AA^n x = 0.$$

The following example shows that it is quite easy to exibit spaces which admits eventually contractive mappings which are not contractive (even are not locally contractive).

Example 2. Let

$$M=\{(x, y): x=\cos t,\ y=\sin t,\ 0<t\leqslant 2\pi\}$$

be the subset of the euclidean plane and let $A: M\to M$ be defined by

$$At=2t, \text{ if } 0<t<\frac{\pi}{2},$$

$$=t+\frac{\pi}{2}, \text{ if } \frac{\pi}{2}\leqslant t<\pi,$$

$$=\frac{t}{2}+\pi, \text{ if } \pi\leqslant t\leqslant 2\pi.$$

In that example A and M satisfy all hypotheses of theorem 1, even of Corollary 1.1, but A is not locally contractive at the point (1,0).

Note that in the above example the following condition, required in [9] and [12], is not satisfied.

There exists a *nbd* $V(p)$ of a unigue fixed point $p\in M$ such that for every *nbd* $U(p)$ there exists $n[U(p)]\in N$ such that

$$A^n[V(p)]\subset U(p) \text{ for all } n\geqslant n[U(p)].$$

The following example serves to show that the hypothesis in Theorem 1, and Corollaries 1.1. and 1.2. do not imply the convergence of $\{A^n x\}_{n\in N}$ for every $x\in M$.

Example 3. Let $M=\{0\}\cup\left\{2, \frac{3}{2}, \ldots, 1+\frac{1}{n}, \ldots\right\}$ and let $A: M\to M$ be defined by $A\left(1+\frac{1}{n}\right)=1+\frac{1}{n+1}$ and $A(0)=0$. Then A is a contractive operator (1) with a fixed point $p=0$, but for every $x\neq 0$ the sequence $\{A^n x\}_{n\in N}$ does not contain a convergent subsequence.

2. In this section we bring sufficient condition for existence of periodic points under eventually ε-contractive operators.

Theorem 2. *Let M be a metric space and A an eventually ε-non-expansive selfmapping of M. If $L(x_0, A)$ is nonempty for some $x_0\in M$ and A is eventually ε-contractive on $L(M, A)$ and orbitally continuous on M, then each $p\in L(x_0, A)$ is a periodic point of A and $0(p, A)=L(x_0, A)$. Furthermore, if $k\in N$ is such that $d(A^n x_0, A^{n+k} x_0)<\varepsilon$ for some $n\in N$, then each $p\in L(x_0, A)$ has a period at most equal to k and* $\lim A^{nk} x_0\in L(x_0, A)$.

Proof. Let p be a point in $L(x_0, A)$ and let $N_1=\{n_i\}\subseteq N$ be such that

$$\lim_i A^{n_i} x_0=p.$$

Hence a subset K of the positive integers N, defined by

$$K=\{r: d(A^n x_0, A^{n+r} x_0)<\varepsilon \text{ for some } n\in N\}$$

is nonempty. Put $k=\min K$ and let $s\in N$ be such that

$$d(A^s x_0, A^{s+k} x_0)<\varepsilon. \tag{15}$$

We shall show that p is a periodic point of a period equal to k, i.e. $A^k(p)=p$. By (15) and eventual ε-nonexpansivity of A

$$(16)\qquad d(A^n x_0, A^{n+k} x_0)=d(A^{n-s} A^s x_0, A^{n-s} A^{s+k} x_0)\leqslant d(A^s x_0, A^{s+k} x_0)<\varepsilon$$

for all $n\geqslant m(A^s x_0, A^{s+k} x_0)+s$. Hence

$$d(A^{n_i} x_0, A^{n_i+k} x_0)\leqslant d(A^s x_0, A^{s+k} x_0)$$

for all sufficiently large $n_i\in N_1$. Then by (10)

$$(17)\qquad d(p, A^k p)=\lim_i d(A^{n_i} x_0, A^{n_i+k} x_0)<\varepsilon.$$

Assume that $A^k p\neq p$. Since $p, A^k p\in L(x_0, A)$, by (17) and (5) there exists an integer $r=m(p, A^k p)$ such that

$$(18)\qquad d(A^r p, A^r A^k p)<d(p, A^k p).$$

By (16) for each fixed $n_i\in N_1$ and $n_i\geqslant m(A^s x_0, A^{s+k} x_0)+s-r$ we have that

$$d(A^{n_i+r} x_0, A^{n_i+r+k} x_0)<\varepsilon.$$

Then by (5)

$$d(A^l x_0, A^{l+k} x_0)=d(A^{l-n_i-r} A^{n_i+r} x_0, A^{l-n_i-r} A^{n_i+r+k} x_0)$$
$$\leqslant d(A^{n_i+r} x_0, A^{n_i+r+k} x_0)<\varepsilon$$

for all $l\geqslant m(A^{n_i+r} x_0, A^{n_i+r+k} x_0)+n_i+r$. Hence we may choose sufficiently large $n_j\in N_1$ such that

$$d(A^{n_j} x_0, A^{n_j+k} x_0)\leqslant d(A^{n_i+r} x_0, A^{n_i+r+k} x_0).$$

Then, as $\{A^{n_j} x_0\}\subset\{A^{n_i} x_0\}$, by (10)

$$\lim_j d(A^{n_j} x_0, A^{n_j+k} x_0)=d(p, A^k p)\leqslant d(A^r p, A^{r+k} p)=$$
$$=\lim_i d(A^{n_i+r} x_0, A^{n_i+r+k} x_0),$$

which is a contradiction with (18). Therefore, we conclude that $A^k p=p$.

Now we shall show that $\lim_n A^{nk} x_0\in L(x_0, A)$ and $L(x_0, A)=\{p, Ap, \ldots, A^{k-1} p\}$. As $\{A^n x_0\}_{n\in N}=\bigcup_{r=0}^{k-1}\{A^{nk-r} x_0\}_{n\in N}$ and $p\in L(x_0, A)$, it follows that p is a cluster point of a subsequence $\{A^{nk-r_0} x_0\}_{n\in N}$ for some r_0, $0\leqslant r_0\leqslant k-1$. Thus, for arbitrary δ, $0<\delta<\varepsilon$, there exists $s\in N$ such that

$$d(p, A^{sk-r_0} x_0)<\delta<\varepsilon.$$

Then, by eventual ε-nonexpansivity of A,

$$d(A^i p, A^i A^{sk-r_0} x_0)\leqslant d(p, A^{sk-r_0} x_0)\ \text{for all}\ i\geqslant m(p, A^{sk-r_0} x_0).$$

Hence, for $i=nk$ we obtain (as $p=A^{nk} p$)

$$d(p, A^{nk} A^{sk-r_0} x_0)<\delta\ \text{for all}\ n\geqslant\frac{1}{k} m(p, A^{sk-r_0} x_0),$$

i.e.

$$p=\lim_n A^{nk} A^{sk-r_0} x_0=\lim_n A^{(n+s)k-r_0} x_0=\lim_n A^{nk-r_0} x_0. \tag{19}$$

Since A is orbitally continuous, it follows from (19) that

$$\lim_n A^{nk+i-r_0} x_0=A^i(\lim_n A^{nk-r_0} x_0)=A^i p\in L(x_0, A);$$

$$i=0, 1, \ldots, r_0, \ldots, k-1.$$

Hence $L(x_0, A)=\{p, Ap, \ldots, A^{k-1}p\}$ and $\lim_n A^{nk} x_0=A^{r_0} p\in L(x_0, A)$.

The proof of the Theorem is complete.

Corollary 2.1. *If M is a compact metric space and A is an orbitally continuous and eventually ε-contractive operator on M, then a set P of periodic points of A is non void and $d(A^s x, A^{s+k} x)<\varepsilon$ for some $s, k\in N$, $x\in M$, implies $\lim_n A^{nk} x=p\in P$ and $A^k p=p$.*

Corollary 2.2. (Edelstein [6], Th. 2.). *Let X be a metric space, f an ε-contractive selfmapping of X satisfying* (14), *then $\xi=\lim_i f^{n_i}(x)$ is a periodic point.*

The following result gives some informations on the set of all periodic points.

Theorem 3. *Let M be a metric space and A an eventually ε-contractive and orbitally continuous selfmappings of M. Then the set P of all periodic points of A is closed and $p, q\in P$, $p\neq q$, implies $d(p, q)\geqslant\varepsilon$. If M is compact, then P is non void and for every $x\in M$ there exists a positive integer k such that $\lim_n A^{nk} x\in P$.*

Proof. Assume that P is non void. Since $p\in P$ implies $p\in L(p, A)\subseteq L(M, A)$, and by Theorem 2 $p\in L(M, A)$ implies $p\in P$, it follows that $P=L(M, A)$.

Let $p, q\in P$ and $0<d(p, q)<\varepsilon$. Then $p=A^k p$ and $q=A^s q$ for some, $k, s\in N$, and by eventual ε-contractivity of A we obtain

$$d(p, q)=d(A^{nks} p, A^{nks} q)<d(p, q) \text{ for all } n\geqslant\frac{1}{k} m(p, q)$$

a contradiction. Thus, $d(p, q)<\varepsilon$ implies $p=q$. Suppose now that p' is a cluster point of P and let $\delta<\varepsilon$ be any positive real number. Then there exists $p\in P$ such that $0<d(p', p)<\frac{\delta}{2}$ and $p=A^k p$ for some $k\in N$. Since A is eventually ε-contractive, it follows that

$$d(A^{nk} p', A^{nk} p)=d(A^{nk} p', p)<d(p', p)<\frac{\delta}{2} \text{ for all } n\geqslant\frac{1}{k} m(p', p)$$

and hence and by $d(p', p)<\frac{\delta}{2}$

$$d(p', A^{nk} p')\leqslant d(p' p)+d(p, A^{nk} p')<\frac{\delta}{2}+\frac{\delta}{2}=\delta \text{ for all } n\geqslant\frac{1}{k} m(p', p).$$

Therefore, $p' \in L(p', A) \subseteq L(M, A) = P$ and $p' = p$ as $d(p', p) < \varepsilon$. Thus, we conclude that P is cluster points free.

If M is compact, then $P = L(M, A)$ is non void, and by above, P is finite. Since $L(x, A)$ is non empty for every $x \in M$, by Theorem 2. there exists $k \in N$ such that $\lim_n A^{nk} x \in L(x, A) \subseteq P$, as asserted.

3. In this section we bring sufficient conditions for the existence of a fixed point under eventually ε-contractive operators.

Theorem 4. *Let M be a metric space, A an eventually ε-contractive and orbitally continuous selfmapping of M and let $L(M, A)$ be non void. If for every $p, q \in L(M, A)$ there exists an ε-chain $p = v_0, v_1, \ldots, v_r = q$ such that.*

$$\sum_{i=1}^{r} d(v_{i-1}, v_i) = \inf \left\{ \sum_{i=1}^{n} d(u_{i-1}, u_i) : p = u_0, u_1, \ldots, u_n = q \quad \text{an } \varepsilon\text{-chain} \right\}, \tag{20}$$

then A has a unique fixed point.

Proof. By Theorem 2 each $p \in L(M, A)$ is a periodic point of A. We shall show that $L(M, A)$ is a singleton.

Assume that $p, q \in L(M, A)$ and $p \neq q$, and let $p = v_0, v_1, \ldots, v_r = q$ be an ε-chain with a property (20). Then $A^j p = p$, $A^l q = q$ for some $j, l \in N$ and hence $A^k p = p$ and $A^k q = q$ for $k = jl$. Since $0 < d(v_{i-1}, v_i) < \varepsilon$ for each $i = 1, 2, \ldots, r$, and A is eventually ε-nonexpansive, for every $v_{i-1}, v_i \in M$ $(i = 1, 2, \ldots, r)$ there exists a positive integer $m(v_{i-1}, v_i) \in N$ such that

$$d(A^n v_{i-1}, A^n v_i) < d(v_{i-1}, v_i) \text{ for all } n \geqslant m(v_{i-1}, v_i) \quad (i = 1, 2, \ldots, r).$$

Put $m = \max\{m(v_0, v_1), m(v_1, v_2), \ldots, m(v_{r-1}, v_r)\}$: Then

$$d(A^{sk} v_{i-1}, A^{sk} v_i) < d(v_{i-1}, v_i) \text{ for all } s \geqslant \frac{1}{k} m \quad (i = 1, 2, \ldots, r).$$

Thence, as $A^{sk} v_0 = A^{sk} p = p$, $A^{sk} v_r = A^{sk} q = q$, the ε-chain

$$p = A^{sk} v_0, A^{ks} v_1, \ldots, A^{sk} v_r = q$$

has a property

$$\sum_{i=1}^{r} d(A^{sk} v_{i-1}, A^{sk} v_i) < \sum_{i=1}^{r} d(v_{i-1}, v_i),$$

which is a contradiction with (20). Therefore, $q = p$ and we proved that $L(M, A) = \{p\}$. Since $p \in L(M, A)$ implies $Ap \in L(M, A)$, we conclude that $Ap = p$, as asserted.

Corollary 4.1. *Let M be a convex metric space and A an eventually ε-contractive and orbitally continuous selfmapping of M. If $L(M, A)$ is non void, then A has a unique fixed point.*

Proof. By convexity of M for every $x, y \in M$ there exists $u \in M$ such that $d(x, u) = d(u, y) = \frac{1}{2} d(x, y)$. Let $0 < d(x, y) < 2\varepsilon$. Then there is $u \in M$ such that $d(x, u) < \varepsilon$, $d(u, y) < \varepsilon$ and $d(x, u) + d(u, y) = d(x, y)$. Let now $2\varepsilon \leqslant d(x, y) < 4\varepsilon$.

Then there exist $u_1, u_2, u_3 \in M$ such that $d(x, u_1) < \varepsilon$, $d(u_1, u_2) < \varepsilon$, $d(u_2, u_3) < \varepsilon$, $d(u_3, y) < \varepsilon$ and $d(x. u_1) + d(u_1, u_2) + d(u_2, u_3) + d(u_3, y) = d(x, y)$. As for every $x, y \in M$ there exists $n(x, y) \in N$ such that $d(x, y) < 2^{n(x, y)} \varepsilon$, by convexity of M there exist an ε-chain $x = v_0, v_1, \ldots, v_2 2^{n(x, y)} = y$ such that

$$\sum_{i=1}^{2^{n(x,y)}} d(v_{i-1}, v_i) = d(x, y)$$

and we may apply Theorem 4.

Theorem 5. *Let M be an ε-chainable metric space, A an orbitally continuous and eventually ε-contractive selfmapping of M satisfying that $L(x_0, A)$ is non void for some $x_0 \in M$. If $p \in L(M, A)$ and $\{A^n x\}_{n \in N}$ has a cluster point whenever $d(p, x) < \varepsilon$, then p is a unique fixed point of A and $p = \lim_n A^n x$ whenever $L(x, A)$ is non void.*

Proof. We shall show that $L(M, A)$ is a singleton $\{p\}$. Suppose that $q \in L(M, A)$ and let $p = v_0, v_1, \ldots, v_r = q$ be an ε-chain from p to q. By Theorem 2, p and q are periodic points of A, consequently there exists $k \in N$ such that $A^k p = p$ and $A^k q = q$. As A is eventually ε-contractive (and we may assume that $v_{i-1} \neq v_i : i = 1, 2, \ldots, r$),

$$d(A^n v_{i-1}, A^n v_i) < d(v_{i-1}, v_i) \text{ for all } n \geqslant m \, (i = 1, 2, \ldots, r),$$

where $m = \max\{m(p, v_1), m(v_1, v_2), \ldots, m(v_{r-1}, q)\}$. Hence

$$(21) \qquad d(A^{jk} v_{i-1}, A^{jk} v_i) < d(v_{i-1}, v_i) \text{ for all } j \geqslant \frac{m}{k} \, (i = 1, 2, \ldots, r).$$

As $d(p, v_1) < \varepsilon$ by assumption $\{A^n v_1\}_{n \in N}$ has a cluster point $p' \in M$. Then p' is a cluster point of $\{A^{jk-s_0} v_1\}_{j \in N}$ for some integer s_0; $0 \leqslant s_0 \leqslant k-1$. Hence, as A is orbitally continuous, $p'' = A^{s_0} p'$ is a cluster point of $\{A^{jk} v_1\}_{j \in N}$, and, as in Theorem 2, $p'' = \lim_j A^{jk} v_1$. Since $p = v_0 = A^{jk} v_0$ for each $j \in N$, by (21) $d(p, p'') < \varepsilon$ and by Theorem 3 $p'' = p$.

Therefore,

$$(22) \qquad p = \lim_j A^{jk} v_1.$$

By (21) for $i = 2$ and by (22) we conclude that $d(p, A^{j_0 k} v_2) < \varepsilon$ for some $j_0 \in N$. Then by assumption $\{A^n A^{j_0 k} v_2\}_{n \in N} \subseteq \{A^n v_2\}_{n \in N}$ has a cluster point in M. Then, as above, we obtain the following relation

$$p = \lim_j A^{jk} v_2$$

which is analogous to the relation (22).

Proceeding in this manner and using that $A^{jk} v_r = A^{jk} q = q$, we obtain

$$p = \lim_j A^{jk} v_1 = \lim_j A^{jk} v_2 = \cdots = \lim_j A^{jk} v_r = q.$$

Therefore, $q \in L(M, A)$ implies $q = p$ and hence $Ap = p$.

Let now $x \in M$ be such that $L(x, A)$ is non void. Then $L(x, A) = \{p\}$ and by Theorem 2 (as $k = 1$)

$$p = \lim_n A^n x,$$

as asserted.

Corollary 5.1. *Let M be an ε-chainable compact metric space and A orbitally continuous and eventually ε-contractive selfmapping of M. Then A has a unique fixed point $p \in M$ and $p = \lim_n A^n x$ for each $x \in M$.*

Corollary 5.2. (Edelstein [6], Th. 3). *Let X be an ε-chainable metric space, f an ε-contractive selfmapping of X satisfying* (14). *If $\xi = \lim_i f^{n_i}(x)$ has a compact spherical neighbourhood $K(\xi, \rho)$ of radius $\rho \geqslant \varepsilon$ then ξ is unique fixed point.*

2. Fixed and periodic points of frequently contractive mappings and mappings with a contractive iteration at a point.

4. In this section we bring sufficient conditions for existence of a unique fixed point under frequently cotractive operators and operators with a contractive iteration at a point.

Theorem 6. *Let M be a metric space and A a frequently non-expansive selfmapping of M. If $\bar{0}(x_0, A)$ is compact for some $x_0 \in M$ and A is an orbitally continuous and frequently contractive operator on $L(M, A)$, then A has a unique fixed point.*

Proof. Since d and the restriction of A to $\bar{0}(x_0, A)$ are continuous, the mapping $B : \bar{0}(x_0, A) \to [0, +\infty)$, defined by $B(x) = d(x, Ax)$, is continuous. As $\bar{0}(x, A)$ is compact by assumption, there is $p \in \bar{0}(x_0, A)$ such that $B(p)$ is infimum of B on $\bar{0}(x_0, A)$. Assume that $B(p) = d(p, Ap) > 0$. If $p \in L(x_0, A)$ then $Ap \in L(x_0, A)$ and, as A is frequently contractive on $L(M, A)$,

$$B(A^{m(p, Ap)} p) = d(A^{m(p, Ap)} p, A^{m(p, Ap)} Ap) < d(p, Ap) = B(p)$$

contradicts the choice of p ($A^{m(p, Ap)} p \in \bar{0}(x_0, A)$ by orbital continuity of A). Therefore, $B(p) = d(p, Ap) = 0$. The uniqueness of the fixed point p follows immediately by frequent contractivity of A on $L(M, A)$.

Now we shall show that $p \in 0(x_0, A)$ and $B(p) = \inf\{B(x) : x \in \bar{0}(x_0, A)\}$, imply $Ap = p$ and hence $p \in L(M, A)$. Suppose that $p \in 0(x_0, A)$. Then by frequent nonexpansivity of A there is a subset $N_1 = \{m_i\}$ of the set N such that

$$d(p, Ap) \geqslant d(A^{m_1} p, A^{m_1} Ap) \geqslant d(A^{m_2} p, A^{m_2} Ap) \geqslant \cdots \geqslant d(A^{m_i} p, A^{m_i} Ap) \geqslant \cdots.$$

As $\{A^{m_i} p\} \subset \bar{0}(x_0, A)$, we may assume that $\{A^{m_i} p\} \to q \in L(x_0, A) \subseteq \bar{0}(x_0, A)$. Then $\{A^{m_i} Ap\} \to Aq$, $d(q, Aq) \leqslant d(p, Ap)$ and consequently $B(q) = B(p)$. Therefore, for the point q we have that $q \in L(x_0, A)$ and $B(q) = \inf\{B(x) : x \in \bar{0}(x_0, A)\}$. This implies, as above, that $B(q) = 0$ and hence $B(p) = 0$. Thus we conclude that $Ap = p$.

The proof of the Theorem is complete.

Corollary 6.1. *Let M be a compact metric space and A a frequently contractive selfmapping of M. If A is orbitally continuous, then A has a unique fixed point.*

Corollary 6.2. (Bailey [2], Cor. 1.2.). *Let M be a compact metric space and A a selfmapping of M satisfying* (2). *Then there exists a unique fixed point in M.*

The following example shows that there are non compact spaces which admit a frequently contractive mapping, not necessarily continuous, and which has a unique fixed point.

Example 4. Let R^2 be the euclidean plane and let

$$E_0=\{(0, 2^{-j}): j=1, 2, 3, \ldots\}\cup\{(0, 0)\},$$

$$E_i=\{(2^{-i}, 2^{-j}): j=1, 2, 3, \ldots, i\}\cup\{(2^{-i}, 0)\},\ i=1, 2, \ldots,$$

$$S=\bigcup_{i=1}^{\infty} E_i \cup E_0.$$

Define A on R^2 as follows:

$$\begin{aligned} A[(x, y)] &= (x, 2y) \text{ if } (x, y)\in S \text{ and } y(2y-1)\neq 0,\\ &= \left(\frac{1}{2}x, 0\right), \text{ if } (x, y)\in S \text{ and } y=\frac{1}{2},\\ &= (x, x), \text{ if } (x, y)\in S \text{ and } y=0,\\ &= \left(\frac{1}{2}, 0\right), \text{ if } (x, y)\in R^2\setminus S \text{ and } y \text{ rational},\\ &= \left(\frac{1}{2}, \frac{1}{2}\right), \text{ if } (x, y)\in R^2\setminus S \text{ and } y \text{ irrational}. \end{aligned}$$

Then A is orbitally continuous and frequently contractive on R^2, and for each $X\in R^2$ the set $0(X, A)$ is compact. The point $0=(0, 0)$ is the unique fixed point of A, and if $X\notin E_0$, then $\lim_n A^n X$ does not exist.

For mappings (8) we have the following result.

Theorem 7. *Let M be a matric space and A a selfmapping of M with a nonexpansive iteration at a point. If $\overline{0}(x_0, A)$ is compact for some $x_0\in M$ and A is an orbitally continuous mapping with a contractive iteration at a point on $L(M, A)$, then A has a unique fixed point $p\in M$ and* $\lim_n A^n x_0=p$.

Proof. By Theorem 6, A has a unique fixed point p in M. To show that $\lim_n A^n x_0=p$, let $\delta>0$. Since p is a cluster point of $0(x_0, A)$, there exists $A^s x_0\in 0(x_0, A)$ such that $d(p, A^s x_0)<\delta$. Then by (8), as $p=Ap=A^{n(p)}p$,

$$\delta>d(p, A^s x_0)\geqslant d(p, A^{n(p)}A^s x_0)\geqslant \cdots \geqslant d(p, A^{in(p)}A^s x_0)\geqslant \cdots$$

Hence

$$\lim_i A^{in(p)}A^s x_0=p.$$

As $\{A^n A^s x_0\}_{n\in N}=\bigcup_{r=1}^{n(p)}\{A^{(i-1)n(p)+r}A^s x_0\}_{i\in N}$ and, by orbital continuity of A, $\lim_i A^{(i-1)n(p)+r}A^s x_0=A^r p=p$ for each $r=1, 2, \ldots, n(p)$, it follows that

$$p=\lim_n A^n A^s x_0=\lim_n A^n x_0,$$

as asserted.

Corollary 7.1. *Let M be a compact metric space and A an orbitally continuous selfmapping of M with a contractive iteration at a point. Then A has a unique fixed point $p \in M$ and $\lim_n A^n x = p$ for each $x \in M$.*

5. In this section we bring sufficient conditions for existence of periodic points of mappings (7) and (9).

Theorem 8. *Let M be a metric space and A a frequently ε-nonexpansive selfmapping of M. If $\overline{0}(x_0, A)$ is compact for some $x \in M$ and A is an orbitally continuous and frequently ε-contractive operator on $L(M, A)$, then a set of periodic points of A is not void.*

Proof. If $0(x_0, A) = \{x_0, Ax_0, A^2 x_0, \ldots\}$ is finite, then x_0 is a periodic point of A. Suppose otherwise and let q be a cluster point of $\{x_0, Ax_0, A^2 x_0, \ldots\}$. Then a subset K of the positive integers N, defined by

$$K = \{r : d(A^n x_0, A^{n+r} x_0) < \varepsilon \text{ for some } n \in N\}$$

is not void. Put $k = \min K$ and let $s \in N$ be such that

$$d(A^s x_0, A^k A^s x_0) = d(A^s x_0, A^{s+k} x_0) < \varepsilon.$$

Since d and the restriction of A^k to $\overline{0}(x_0, A)$ are continuous, the mapping $B : \overline{0}(x_0, A) \to [0, +\infty)$, defined by $B(x) = d(x, A^k x)$ is continuous. As $\overline{0}(x_0, A)$ is compact, there exists $p \in \overline{0}(x_0, A)$ such that

$$B(p) = \text{int}\{B(x) : x \in \overline{0}(x_0, A)\}.$$

Then $B(p) \leqslant B(A^s x_0) < \varepsilon$. Assume that $B(p) > 0$. If $p \in L(x_0, A)$, then $A^k p \in L(x_0, A)$ by orbital continuity of A. Then $0 < B(p) = d(p, A^k p) < \varepsilon$ and frequent ε-contractivity of A imply that

$$B(A^{m(p, A^k p)} p) = d(A^{m(p, A^k p)} p, A^{m(p, A^k p)} A^k p) < d(p, A^k p) = B(p).$$

This contradicts the choice of p. Therefore, $B(p) = 0$ and hence $A^k p = p$.

If $p \in 0(x_0, A)$, then $B(p) = d(p, A^k p) < \varepsilon$ and frequent ε-nonexpansivity of A imply that there exists a subset $\{m_i\}$ of N such that

$$d(p, A^k p) \geqslant d(A^{m_1} p, A^{m_1} A^k p) \geqslant \cdots \geqslant d(A^{m_i} p, A^{m_i} A^k p) \geqslant \cdots.$$

As $\{A^{m_i} p\}_{n \in N} \subseteq \overline{0}(x_0, A)$, we may assume that $\{A^{m_i} p\} \to p' \in L(x_0, A)$. Then $\{A^{m_i} A^k p\} \to A^k p'$, $d(p', A^k p') \leqslant d(p, A^k p)$ and consequently $B(p') = B(p)$. Therefore, we again have $p' \in L(x_0, A)$ and $B(p') = \inf\{B(x) : x \in \overline{0}(x_0, A)\}$, which imply $B(p') = 0$. Thus, $B(p) = B(p') = 0$ and we conclude that $A^k p = p$, as asserted.

Corollary. 8.1. *Let M be a compact metric space and A a frequently ε-contractive selfmapping of M. If A is orbitally continuous, then a set of periodic points of A is non void and finite.*

Corollary 8.2. (Bailey [2] *Cor.* 2.2.). *Let M be a compact metric space and A a selfmapping of M satisfying* (3). *Then there exists a non-null finite set of periodic points.*

Corollary 8.3. *Let M be a metric space and let A be a selfmapping of M with an ε-nonexpansive iteration at a point. If $\overline{O}(x_0, A)$ is compact for some $x_0 \in M$ and A is an orbitally continuous mapping with ε-contractive iteration at a point on $L(M, A)$, then a set of periodic points of A is not void.*

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