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ON SPECTRAL STPUCTURE OF GRAPHS HAVING THE MAXIMAL EIGENVALUE NOT GREATER THAN TWO

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Let G be a graph with n vertices and the adjacency matrix A. The polynomial

$$P_G(\lambda) = \det(\lambda I - A) = \lambda^n + a_1 \lambda^{n-1} + \cdots + a_n \tag{1}$$

is called the characteristic polynomial of G. The roots $\lambda_1, \lambda_2, \ldots, \lambda_n$ of $P_G(\lambda) = 0$ are called the eigenvalues of G. The "system of numbers" $[\lambda_1, \lambda_2, \ldots, \lambda_n]$ is called the spectrum of the graph G and will be denoted also by **G**. Note that some λ_i can be mutually equal.

Since A is a symmetric matrix, the eigenvalues of G are real. The maximal number in the spectrum of G is called the index of the graph G and is denoted by $\Lambda(G) = \Lambda$.

Relations between the structure of a graph and its index were the topic of several investigations [1 – 3]. In particular, J. H. Smith [1] was able to determine the set $\mathscr{S}$ of all graphs having $\Lambda \leqslant 2$. The connected graphs from $\mathscr{S}$ are shown in Fig 1.

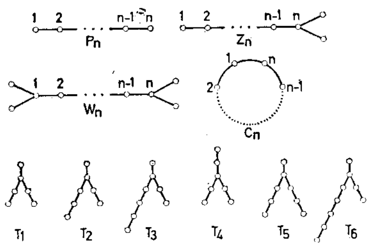

Fig. 1

In this paper we shall study the properties of the spectra of graphs from $\mathscr{S}$. It will be shown that the condition $\Lambda \leqslant 2$ demands a particular algebraic form of the eigenvalues of these graphs. In addition, a procedure for deciding whether a system of numbers is a spectrum of a graph from $\mathscr{S}$ or not will be described. Moreover, all graphs, having the spectrum equal to a given system of numbers, can be determined by this procedure.

The spectrum of the union of two graphs (for definitions not given here see [4]) is obviously the union of their spectra (having in view the multiplicities of the eigenvalues). The expression G_1+G_2 will denote the union of the graphs G_1 and G_2 and $\mathbf{G}_1+\mathbf{G}_2$ the union of their spectra. kG ($k\,\mathbf{G}$) denotes the union of k copies of $G(\mathbf{G})$. If $\mathbf{G}_2\subset\mathbf{G}_1$, the expression $\mathbf{G}_1-\mathbf{G}_2$ denotes the difference of systems of numbers $\mathbf{G}_1$ and $\mathbf{G}_2$.

The spectra of graphs from Fig. 1 can be easily obtained and so we have:

$$\mathbf{P}_n=\left[2\cos\frac{j\pi}{n+1}\,\middle|\,j=1,\,2,\,\ldots,\,n\right],$$

$$\mathbf{Z}_n=\left[2\cos\frac{(2j+1)\pi}{2(n+1)}\,\middle|\,j=0,\,1,\,\ldots,\,n\right]+[0],$$

$$\mathbf{W}_n=\left[2\cos\frac{j\pi}{n+1}\,\middle|\,j=1,\,2,\,\ldots,\,n\right]+[2,\,0,\,0,\,-2],$$

$$\mathbf{C}_n=\left[2\cos\frac{2j\pi}{n}\,\middle|\,j=1,\,2,\,\ldots,\,n\right],$$

$$\mathbf{T}_1=\left[2\cos\frac{j\pi}{12}\,\middle|\,j=1,\,4,\,5,\,7,\,8,\,11\right],$$

$$\mathbf{T}_2=\left[2\cos\frac{j\pi}{18}\,\middle|\,1=1,\,5,\,7,\,9,\,11,\,13,\,17\right],$$

$$\mathbf{T}_3=\left[2\cos\frac{j\pi}{30}\,\middle|\,j=1,\,7,\,11,\,13,\,17,\,19,\,23,\,29\right],$$

$$\mathbf{T}_4=\left[2\cos\frac{2j\pi}{6}\,\middle|\,j=1,\,2,\,3,\,4,\,5,\,6\right]+[0],$$

$$\mathbf{T}_5=\left[2\cos\frac{j\pi}{4}\,\middle|\,j=1,\,2,\,3\right]+[2,\,1,\,0,\,-1,\,-2],$$

$$\mathbf{T}_6=\left[2\cos\frac{j\pi}{5}\,\middle|\,j=1,\,2,\,3,\,4\right]+[2,\,1,\,0,\,-1,\,-2].$$

Spectra of the path P_n and the circuit C_n are well known [5], while the spectra of Z_n and W_n can be easily obtained and are given in [6] and [7]. The validity of the given spectra for T_1-T_6 can be checked by direct calculation, although this is not simple in all cases.

From the quoted facts the following theorem is evident.

Theorem 1. 1° *All eigenvalues of any graph from* $\mathcal{S}$ *are of the form* $2\cos\frac{p}{q}\pi$, *where* p *and* q *are integers and* $q\neq 0$. *The index of a graph from* $\mathcal{S}$ *is either equal to 2 or is of the form* $2\cos\frac{\pi}{q}$ *where* q *is a natural number. Only graphs from* $\mathcal{S}$ *have these two properties.*

2° The set of eigenvalues of all graphs from $\mathcal{S}$ is equal to the set of all numbers of the form $2\cos\frac{p}{q}\pi$, *where p and q are integers and* $q \neq 0$.

Since $|2\cos\theta| \leqslant 2$, it is clear that the result which is summarized in Theorem 1 is intimately related to the condition $\Lambda \leqslant 2$. It is to be expected that in the spectra of graphs with the property $\Lambda \leqslant 2+\varepsilon$ for any $\varepsilon > 0$, numbers are contained having a different (probably much more complicated) algebraic form. In connection to this we mention that graphs P_n, Z_n and W_n are the only three graphs of the form from Fig. 2 the spectrum of which is known in a closed

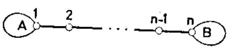

Fig. 2

analytical form. According to the results obtained by Hoffman [2], the spectrum of the graph from Fig. 3 would be of a special interest.

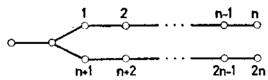

Fig. 3

If graphs G_1 and G_2 have the same spectra, we shall write $\mathbf{G}_1 = \mathbf{G}_2$. Thus G_1 and G_2 are isospectral. A pair of isospectral nonisomorhpic graphs will be called a PING. Now we shall list some PING's from the set $\mathcal{S}$. As it will be shown later, the following relations enable the generation of all PING's contained in $\mathcal{S}$. We have

(2a) $\mathbf{Z}_n + \mathbf{P}_n = \mathbf{P}_{2n+1} + \mathbf{P}_1$,

(2b) $\mathbf{W}_n = \mathbf{C}_4 + \mathbf{P}_n$,

(2c) $\mathbf{C}_{2n} + 2\,\mathbf{P}_1 = \mathbf{C}_4 + 2\,\mathbf{P}_{n-1}$,

(2d) $\mathbf{T}_1 + \mathbf{P}_3 + \mathbf{P}_5 = \mathbf{P}_1 + \mathbf{P}_2 + \mathbf{P}_{11}$,

(2e) $\mathbf{T}_2 + \mathbf{P}_5 + \mathbf{P}_8 = \mathbf{P}_{17} + \mathbf{P}_2 + \mathbf{P}_1$,

(2f) $\mathbf{T}_3 + \mathbf{P}_{14} + \mathbf{P}_9 + \mathbf{P}_5 = \mathbf{P}_{29} + \mathbf{P}_4 + \mathbf{P}_2 + \mathbf{P}_1$,

(2g) $\mathbf{T}_4 + \mathbf{P}_1 = \mathbf{C}_4 + 2\,\mathbf{P}_2$,

(2h) $\mathbf{T}_5 + \mathbf{P}_1 = \mathbf{C}_4 + \mathbf{P}_3 + \mathbf{P}_2$,

(2i) $\mathbf{T}_6 + \mathbf{P}_1 = \mathbf{C}_4 + \mathbf{P}_4 + \mathbf{P}_2$.

For $n=1$ relation (2b) yields the PING shown in Fig. 4. This is the unique PING with graphs with 5 vertices. There is no PING with graphs with less than 5 vertices [8].

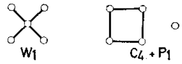

Fig. 4

Let $\mathbf{S}_1, \mathbf{S}_2, \ldots, \mathbf{S}_m$ be some systems of numbers and $\sigma_1, \sigma_2, \ldots, \sigma_m$ integers such that the expression

$$\sigma_1 \mathbf{S}_1 + \sigma_2 \mathbf{S}_2 + \cdots + \sigma_m \mathbf{S}_m \tag{3}$$

can be calculated in at least one way by successive performing the quoted operations. Then (3) defines a systems $\mathbf{S}$ and we shall say that $\mathbf{S}$ is a linear combination of $\mathbf{S}_1, \mathbf{S}_2, \ldots, \mathbf{S}_m$.

T h e o r e m 2. *The spectrum of any bipartite graph from $\mathcal{S}$ can be represented in a unique way as a linear combination of the form*

$$\sigma_0 \mathbf{C}_4 + \sigma_1 \mathbf{P}_1 + \sigma_2 \mathbf{P}_2 + \cdots + \sigma_m \mathbf{P}_m . \tag{4}$$

The number m is bounded by a function of the number of vertices. σ_0 is always non-negative and the non-vanishing coefficient σ_i with the greatest i is positive.

P r o o f. From the relations (2) we see that the spectrum of each connected bipartite graph from $\mathcal{S}$ can be expressed as a linear combination of $\mathbf{C}_4, \mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3, \ldots$. Since the spectrum of a disconnected graph is a linear combination of spectra of some connected graphs, the last statement can be immediately extendend to every bipartite graph from $\mathcal{S}$. For example, we have

$$\mathbf{Z}_n = \mathbf{P}_{2n-1} - \mathbf{P}_n + \mathbf{P}_1,$$

$$\mathbf{T}_3 = \mathbf{P}_{29} - \mathbf{P}_{14} - \mathbf{P}_9 - \mathbf{P}_5 + \mathbf{P}_4 + \mathbf{P}_2 + \mathbf{P}_1 .$$

There exists no PING whose graphs contain only paths and circuits C_4 as components. This follows from the fact that no two graphs $\mathbf{C}_4, \mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3, \ldots$ have the same index. According to this, two linear combinations $\sigma_0 \mathbf{C}_4 + \sigma_1 \mathbf{P}_1 + \sigma_2 \mathbf{P}_2 + \cdots + \sigma_m \mathbf{P}_m$ and $\sigma'_0 \mathbf{C}_4 + \sigma'_1 \mathbf{P}_1 + \sigma'_2 \mathbf{P}_2 + \cdots + \sigma'_m \mathbf{P}_m$ are equal if, and only if $\sigma_0 = \sigma'_0, \sigma_1 = \sigma'_1, \sigma_2 = \sigma'_2, \ldots, \sigma_m = \sigma'_m$.

This completes the proof of the Theorem 2.

One of the fundamental problems in the spectral graph theory is the following one. Let a system of numbers be given. The question arises whether there exists a graph (from a given class of graphs) or not, the spectrum of which is equal to the given system of numbers.

Of course, for a given system $\mathbf{S}$, containing n numbers one can check all graphs with n vertices in order to answer the above question. Nevertheless, we would like to present a more effective procedure which enables the determination of all graphs having the spectrum equal to a given system of numbers of the form $2 \cos \frac{p}{q} \pi$.

Let first consider only bipartite graphs from $\mathcal{S}$. As it is well known, bipartite graphs have a symmetric spectrum with respect to the zero point. Given a symmetric system $\mathbf{S}$ of numbers* of the form $2 \cos \frac{p}{q} \pi$, we try to represent it as a linear combination of $\mathbf{C}_4, \mathbf{P}_1, \mathbf{P}_2, \ldots$. If this is not possible,

* This means that if a number α belongs to $\mathbf{S}$ with the multiplicity p, then $-\alpha$ also belongs to $\mathbf{S}$ with the same multiplicity p.

S is not a spectrum of any graph (according to Theorems 1 and 2). In the case such a representation is possible, the mentioned linear combination is unique. Principles of finding the corresponding coefficients are clear since among $\mathbf{C}_4, \mathbf{P}_1, \mathbf{P}_2, \ldots$ no two systems have the same greatest element.

Let now **S** be represented as

$$\mathbf{S} = \sigma_0 \mathbf{C}_4 + \sigma_1 \mathbf{P}_1 + \sigma_2 \mathbf{P}_2 + \cdots + \sigma_m \mathbf{P}_m. \tag{5}$$

Suppose that **S** is the spectrum of a graph. Presenting it as a linear combination of spectra of the components we get

$$\begin{aligned}\mathbf{S} = {} & p_1 \mathbf{P}_1 + p_2 \mathbf{P}_2 + p_3 \mathbf{P}_3 + \cdots + z_1 \mathbf{Z}_1 + z_2 \mathbf{Z}_2 + z_3 \mathbf{Z}_3 + \cdots \\ & + w_1 \mathbf{W}_1 + w_2 \mathbf{W}_2 + w_3 \mathbf{W}_3 + \cdots \quad + c_2 \mathbf{C}_4 + c_3 \mathbf{C}_6 + \cdots \\ & + t_1 \mathbf{T}_1 + t_2 \mathbf{T}_2 + t_3 \mathbf{T}_3 + t_4 \mathbf{T}_4 + t_5 \mathbf{T}_5 + t_6 \mathbf{T}_6\end{aligned} \tag{6}$$

for some non-negative integers

$$\begin{aligned}& p_1, p_2, p_3, \ldots, z_1, z_2, z_3, \ldots, w_1, w_2, w_3, \ldots, c_2, c_3, \ldots, \\ & t_1, t_2, t_3, t_4, t_5, t_6.\end{aligned} \tag{7}$$

Using the relations (2) one can express the eq. (6) in the form

$$\mathbf{S} = F_0 \mathbf{C}_4 + F_1 \mathbf{P}_1 + F_2 \mathbf{P}_2 + \cdots, \tag{8}$$

where the coefficients $F_i (i = 0, 1, \ldots)$ in (8) are functions of variables (7). Hence,

$$\begin{aligned}F_0 = {} & (w_1 + w_2 + w_3 + \cdots) + (c_2 + c_3 + \cdots) + \\ & + t_4 + t_5 + t_6\end{aligned} \tag{9a}$$

$$\begin{aligned}F_1 = {} & p_1 - z_1 + w_1 + 2c_2 + (z_1 + z_2 + z_3 + \cdots) - \\ & - 2(c_2 + c_3 + \cdots) + t_1 + t_2 + t_3 - t_4 - t_5 - t_6\end{aligned} \tag{9b}$$

and for $i > 1$ and $i \neq 2, 3, 4, 5, 8, 9, 11, 14, 17, 29$

$$F_i = \tilde{F}_i, \tag{9c}$$

where

$$\tilde{F}_i = \begin{cases} p_i \qquad\quad - z_i + w_i + 2c_{i+1}, & i \text{ even} \\ p_i + z_{\frac{i-1}{2}} - z_i + w_i + 2c_{i+1}, & i \text{ odd} \end{cases} \tag{9d}$$

for the excluded values of i

$$F_i = \tilde{F}_i + h_i, \tag{9e}$$

where

$$\begin{aligned}& h_2 = t_1 + t_2 + t_3 + 2t_4 + t_5 + t_6, \\ & h_3 = -t_1 + t_5, \quad h_4 = t_3 + t_6, \\ & h_5 = -t_1 - t_2 - t_3, \quad h_8 = -t_2, \quad h_9 = -t_3, \\ & h_{11} = t_1, \quad h_{14} = -t_3, \quad h_{17} = t_2, \quad h_{29} = t_3.\end{aligned} \tag{9f}$$

Comparing (5) and (8) we get the following system of linear algebraic equations in unknowns (7):

(10) $$F_i = \sigma_i \qquad i = 0, 1, 2, \ldots, m.$$

Now we can formulate the following theorem.

Theorem 3. *Let* **S** *be a symmetric system of numbers of the form* $2\cos\frac{p}{q}\pi$ *where* p, q, *are integers and* $q \neq 0$. *A necessary condition for* **S** *to be a graph spectrum is that* **S** *can be represented in the form* (5). *In this case, to every solution of the system of equations* (10) *in unknowns* (7), *these quantities being non-negative integers, a graph corresponds, the spectrum of which is* **S**. *All graphs having the spectrum equal to* **S** *can be obtained in this way.*

If **S** is not symmetric and is the spectrum of a graph, this graph necessarily contains some circuits of odd lengths as components. According to a theorem due to H. Sachs [9], the length of the shortest odd circuit in a graph is equal to the index f of the first non-vanishing coefficient among $a_1, a_3, a_5, \ldots$ from the correspoding characteristic polynomial (1) and the number of shortest odd circuits is equal to $-a_f/2$.

Hence, we have to determine the polynomial whose zeros are the numbers from **S**. The coefficients of this polynomial must be integers. If so, we find f and $-a_f/2 = k$, (k must be an integer), and **S** must contain k times the spectrum of a circuit C_f. If this is not true, **S** is not a graph spectrum. In the other case we remove from **S** all eigenvalues belonging to k circuits C_f and get a new system **S**$'$. If **S**$'$ is symmetric, we apply Theorem 3. If not, the above described procedure is to be applied on **S**$'$ etc.

Theorem 4. *Let* **S** *be a non-symmetric system of numbers of the form* $2\cos\frac{p}{q}\pi$, *where* p, q *are integers and* $q \neq 0$. *A necessary condition for* **S** *to be a graph spectrum is that* **S** *can be represented as a union of a symmetric system and a linear combination of spectra of some circuits of odd lengths. This linear combination (if it exists) can be uniquely determined from* **S**.

Theorems 3 and 4 solve entirely the problem of deciding whether or not is a given system of numbers of the form $2\cos\frac{p}{q}\pi$ equal to the spectrum of some graph. Moreover, using these theorems all graphs having the spectrum equal to **S** can be determined

Corollary. *All* PING's *from* $\mathcal{S}$ *can be generated using relation* (2).

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