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ELEMENTARY TAUBERIAN THEOREMS FOR REGULAR LINEAR OPERATORS

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Let $\mathcal{F}$ denote the set of real-valued functions on $\mathbf{R}^+=\{t:t\geqslant 0\}$ which are bounded on every finite interval, and let $\mathcal{L}$ denote some linear subspace and sublattice of $\mathcal{F}$. (The order relation in $\mathcal{F}$, $f\leqslant g$, is defined by $f(t)\leqslant g(t)\ \forall\ t\in\mathbf{R}^+$.)

Let A be a linear operator defined on $\mathcal{L}$ with values in $\mathcal{F}$. It will be convenient to use the following notations. If $g=A(f)$, then $A(f, x)$ or $A(f(t), x)$ will denote the value at x of the function g. If $f(t, x)$ is a function in $\mathcal{L}$ for each fixed value of $x\geqslant 0$, and g_x is its image by operator A, then $A(f(t, x), x)$ will denote the value of g_x at x.

A linear operator A is called *regular* on $\mathcal{L}$ if one of the following properties, which are equivalent, is satisfied.

(i) $$A(f)=A_1(f)-A_2(f),\quad \forall f\in\mathcal{L},$$

where A_i are two positive linear operators on $\mathcal{L}$, i.e.

$$f,\ g\in\mathcal{L},\ f\leqslant g\ \Rightarrow\ A_i(f)\leqslant A_i(g);$$

(ii) $$V_A(f, x)<\infty,\quad \forall x\in\mathbf{R}^+,\quad \forall f\in\mathcal{L},$$

where

$$V_A(f, x)=\sup\{|A(g, x)|:g\in\mathcal{L},\ |g|\leqslant|f|\}.$$

Regular operators on partially ordered linear spaces have bee studied in [1], [2] and [3]. A proof of equivalence of the two possible definitions is sketched in [4]. Most common examples of regular operators are integral transforms of functions

(1.1) $$A(f)=\int_0^\infty K(x, t)f(t)\,\mathrm{dt}$$

where

$$V_A(f, x)=\int_0^\infty |K(x, t)|\,|f(t)|\,dt<\infty\ \forall\ x\in\mathbf{R}^+,$$

or (when $\mathscr{L}$ is replaced by the set of all real-valued sequences) matrix transforms of sequences

$$A(s) = \sum_{k=0}^{\infty} a_k(x) s_k \tag{1.2}$$

where

$$V_A(s, x) = \sum_{k=0}^{\infty} |a_k(x)| |s_k| < \infty, \quad \forall x \in \mathbf{R}^+.$$

It is a natural problem to investigate whether classical results on asymptotic behavior of transforms of type (1.1) and (1.2) can be extended to regular linear operators. In [4] the authors have obtained generalizations of classical Abelian theorems, two of which are quoted here (Th. I, § 2 and Th. II, § 3). In this paper, two typical Tauberian theorems, one relative to the convergence of functions at infinity, the second relative to slowly varying functions, will be established (Th. 1, § 2 and Th. 2, § 3); in some sense, they are converse of the theorems established in [4].

2. A Tauberian Theorem for Convergence

In [4], the authors have proved the following result, which is an extension of the theorem of Toeplitz and Schur for transforms (1.2) [5, Th. 2, p. 43] and of Raff for transforms (1.1) [6].

Theorem I. *Let A be a regular operator on the space $\mathscr{M} \subset \mathscr{F}$ of measurable functions. Then,*

$$f(x) \to c\ (x \to \infty) \Rightarrow A(f, x) \to c\ (x \to \infty)$$

if and only if the following three conditions are satisfied:

$$A(1, x) \to 1\ (x \to \infty), \tag{2.1}$$

$A(\chi_E(t), x) = o(1)\ (x \to \infty)$ *for every bounded measurable set* $E \subset \mathbf{R}^+$, *and*

$$W_A(1, x) = O(1) \quad (x \to \infty), \tag{2.2}$$

where

$$W_A(f, x) = \sup\{|A(g, x)| : g \in M,\ |g| \leqslant |f|,\ g(x) = o(f(x))\ (x \to \infty)\}. \tag{2.3}$$

Theorem 1 below gives a converse of this theorem. At the same time, it generalizes Tauberian theorems for special transforms of the form (1.1) given by J. Karamata and G. E. Peterson, which will be quoted first.

In [7, Th. A p. 14; see also Th. I p. 26], J. Karamata has proved the following.

Let λ be a continuous monotone function on $\mathbf{R}^+$ such that $\lambda(x) \to \infty (x \to \infty)$, and let $y = y(x)$ be

(2.4) continuous, monotone on $\mathbf{R}^+$ and $y(x) \to \infty (x \to \infty)$.

Suppose

$$K(x, t) \geqslant 0, \quad \int_0^{\infty} K(x, t)\, dt = 1, \quad \int_0^{M} K(x, t)\, dt = o(1) \quad (x \to \infty)$$

for every $M>0$, and

$$\int_0^\infty K(x,t)\left|\log\frac{\lambda(t)}{\lambda(y)}\right|dt=O(1)\quad(x\to\infty). \tag{2.5}$$

If f is of bounded variation on every finite interval of R^+, then

$$\int_0^\infty K(x,t)f(t)\,dt=o(1)\ (x\to\infty)\ \Rightarrow\ f(x)=o(1)\quad(x\to\infty) \tag{2.6}$$

whenever

$$\frac{1}{\lambda(x)}\int_0^x \lambda(t)\,df(t)=o(1)\quad(x\to\infty). \tag{2.7}$$

Peterson's result can be stated as follows.

Let φ be such that

$$\begin{cases}\varphi(x)\ \text{is non decreasing on}\ R^+,\\ \dfrac{\varphi(x)}{\varphi(x/2)}=O(1)\quad(x\to\infty),\\ \Phi(x)=\displaystyle\int_0^x\frac{dt}{\varphi(t)}\ \text{exists for all}\ x>0,\\ \varphi(x)\Phi(x)=O(x)\quad(x\to\infty).\end{cases} \tag{2.8}$$

Let ρ be an increasing differentiable function which maps $[c,\infty)$ onto $[0,\infty)$. $c\geqslant 0$. Let also y be a function that satisfies the conditions

$$\begin{cases}y(x)\to\infty\ (x\to\infty),\\ \forall M>0,\ \exists K>0\ \text{such that}\ (K,\infty)\subset y([M,\infty)).\end{cases} \tag{2.9}$$

Suppose that transform (1.1) carries functions converging to a finite limit into functions converging to the same limit, and suppose that

$$\sigma^2(x)=\int_c^\infty(\rho(t)-\rho(y))^2\,|K(x,t)|\,dt$$

exists for every large x and

$$\sigma(x)=O(\varphi(y))\quad(x\to\infty). \tag{2.10}$$

If

$$f(x)=\int_0^x a(t)\,dt,$$

then, (2.6) is true whenever

$$a(x)=o\left(\frac{\rho'(x)}{\varphi(\rho(x))}\right)\quad(x\to\infty). \tag{2.11}$$

Karamata's condition (2.7) is more general than Peterson's condition (2.11). However, Karamata's hypothesis that $K(x, t) \geqslant 0$ and condition (2.5) are clearly stronger than condition (2.10).

In all applications of either theorem, the function $y(x)$ defined by condition (2.4) or (2.9) is replaced by $y(x)=x$. Also, the proofs of the theorems are essentially the same, whether $y(x)$ satisfies one of the more general conditions or is equal to x. As a consequence, the case $y(x)=x$ will be prefered in this paper. Except for this simplification, the following result for arbitrary regular operators contains both Karamata's and Peterson's theorems as special cases.

Theorem 1. *Let φ satisfy condition* (2.8) *and let ρ be differentiable on* $\mathbf{R}^+$, $\rho(t)=0$ *on* $[0, t_0]$, $\rho'(t)>0$ *for* $t>t_0$, *and* $\rho(t)\to\infty$ $(t\to\infty)$. *Suppose that A is a regular linear operator satisfying the conditions of Theorem* I, *with W_A defined as in* (2.3). *Suppose also that*

$$W_A(\rho(t)-\rho(x), x)=O(\varphi(\rho(x))) \quad (x\to\infty). \tag{2.12}$$

If f is of bounded variation on every finite interval of $\mathbf{R}^+$, *then*

$$A(f, x)\to c \ (x\to\infty) \ \Rightarrow \ f(x)\to c\,(x\to\infty)$$

whenever

$$\frac{1}{\lambda(x)}\int_0^x \lambda(t)\,df(t)=o(1) \quad (x\to\infty) \tag{2.13}$$

where

$$\lambda(x)=\exp(\Phi(\rho(x))). \tag{2.14}$$

The Tauberian condition (2.13) is satisfied whenever (2.11) holds. Indeed, from the properties of φ follows that

$$\Phi(x)\nearrow\infty\,(x\to\infty) \tag{2.15}$$

(see [8, proof of Lemma 2]), so that $\lambda(x)\nearrow\infty$ $(x\to\infty)$. On the other hand, (2.11) and (2.14) imply that

$$\frac{1}{\lambda(x)}\int_0^x \lambda(t)\,df(t)=\frac{1}{\lambda(x)}\int_0^x \varepsilon(t)\,\lambda'(t)\,dt,$$

where $\varepsilon(x)=o(1)$ $(x\to\infty)$. From those two statements, (2.13) follows. Furthermore (2.13) cannot be improved in the sense that it is already a necessary condition for f to tend to a limit (see [7, p. 15—16]).

The theorems of Karamata and Peterson remain true if "o" is replaced by "O" in the conditions (2.7) or (2.11) as well as in the conclusion (2.6). Theorem 1 can be modified in a similar way. In this new version, $A(f, x)=O(1)$ $(x\to\infty)$ implies $f(x)=O(1)$ $(x\to\infty)$ whenever condition (2.13) is satisfied with "o" replaced by "O". The proof of those results is similar to the proof of the corresponding "o"-theorems.

3. A Tauberian Theorem for Slowly Varying Functions

A function f on $\mathbf{R}^+$ is said to be *slowly varying* if it is measurable, positive, bounded away from 0 and ∞ in every finite interval and satisfies the condition

$$\frac{f(\lambda x)}{f(x)} \to 1 \ (x \to \infty), \ \forall \lambda > 0. \tag{3.1}$$

The class $\mathcal{S}$ of slowly varying functions has been been introduced by Karamata [9], [10] (see also [11; chap. 7, § 8, 9]).

Since $\mathcal{S}$ is an extension of the class of functions converging to a finite positive limit at infinity, a natural problem is to find Abelian and Tauberian theorems relative to the class $\mathcal{S}$. An Abelian theorem has been established in [12] and [13] for transforms of the form (1.1) and (1.2), and has been generalized to regular operators in [4]. The result is quoted below. A Tauberian-type result is established in Theorem 2 for regular operators; it generalizes a similar result obtained by H. Baumann [14] for special transforms (1.2).

It will be convenient to introduce the notation

$$\mathbf{t}^\eta = \begin{cases} \min\{1, t^\eta\}, & \text{for } 0 \leqslant t \leqslant 1, \\ t^\eta, & \text{for } t > 1, \end{cases}$$

and to make use of the asymptotic relation $f(x) \simeq g(x)$ $(x \to \infty)$, which means $f(x)/g(x) \to 1$ $(x \to \infty)$.

The result proved in [4] may be stated as follows.

Theorem II. *Let A be a regular operator on the linear space $\mathcal{M} \subset \mathcal{F}$ of measurable functions. Then $A(f, x) \simeq f(x)$ $(x \to \infty)$ for every slowly varying function f if and only if the following conditions are satisfied*

$$W_A(\mathbf{t}^\eta, x) = O(x^\eta) \quad (x \to \infty), \tag{3.2}$$

for all η in some interval $(-\eta_0, \eta_0)$, and

$$A(1, x) \to 1 \quad (x \to \infty). \tag{3.3}$$

A converse of Theorem II is proved here in the following form.

Theorem 2. *Let A be a regular operator satisfying the condition of Theorem* II. *If f is a strictly positive, measurable function which is bounded in every finite interval and s is a slowly varying function, then*

$$A(f, x) \simeq s(x) (x \to \infty) \ \Rightarrow \ f(x) \simeq s(x) \quad (x \to \infty) \tag{3.4}$$

whenever

$$\max_{x \leqslant t \leqslant \lambda x} \frac{|f(t) - f(x)|}{s(x)} = o(1) \quad (x \to \infty) \tag{3.5}$$

for some $\lambda > 1$.

Note that condition (3.5) is necessary in order that statement (3.4) be true. Indeed, this last relation implies that $f\in\mathcal{S}$, and a well-known property of functions of $\mathcal{S}$ is that limit (3.1) is uniform with respect to λ in any finite subinterval of $(0, \infty)$; thus

$$\max_{x\leqslant t\leqslant\lambda x}\left|\frac{f(t)}{f(x)}-1\right|\to 0 \quad (x\to\infty)$$

and (3.5) follows.

Regularly varying functions are defined as functions of the form $x^{\sigma}s(x)$, with $s\in\mathcal{S}$ and $\sigma\in\mathbf{R}$ (see [9], [10]). Let $\mathcal{R}_{\sigma}$ denote the class of such functions for a fixed value of σ. It is easy to extend the results of Theorem II and Theorem 2 to the class $\mathcal{R}_{\sigma}$. Indeed, let A be a regular linear operator and define $B(f(t), x)=x^{-\sigma}A(t^{\sigma}f(t), x)$. Operator B is again a regular linear operator, and it will carry functions of $\mathcal{S}$ into asymptotically equivalent functions whenever operator A transforms functions of $\mathcal{R}_{\sigma}$ into asymptotically equivalent ones. This leads to the following results.

Let A be a regular operator on the linear space $\mathcal{M}\subset\mathcal{F}$ of measurable functions. Then $A(f, x)\simeq f(x)\,(x\to\infty)$, for every $f\in\mathcal{R}_{\sigma}$, if and only if

$$A(\mathbf{t}^{\sigma}, x)\simeq x^{\sigma} \text{ and } W_A(\mathbf{t}^{\eta}, x)=O(x^{\eta}) \quad (x\to\infty)$$

for all η in some interval $(\sigma-\eta_0,\ \sigma+\eta_0)$.

Conversely, suppose operator A satisfies the preceding conditions. Then, if $s\in\mathcal{S}$,

$$A(f, x)\simeq x^{\sigma}s(x)\,(x\to\infty) \Rightarrow f(x)\simeq x^{\sigma}s(x) \quad (x\to\infty)$$

whenever f satisfies the Tauberian condition

$$\max_{x\leqslant t\leqslant\lambda x}\frac{|f(t)\,t^{-\sigma}-f(x)\,x^{-\sigma}|}{s(x)}=o(1) \quad (x\to\infty).$$

Particular cases of these statements for matrix transforms of sequences are found in [13] and [14].

Proof of Theorem 1. In order to prove that $f(x)\to c$ $(x\to\infty)$, it is sufficient to show that

$$|A(f, x)-f(x)A(1, x)|=o(1) \quad (x\to\infty). \tag{4.1}$$

Indeed, the assertion will then follow from (2.1) and the hypothesis that $A(f, x)\to c$ $(x\to\infty)$.

Let

$$\delta(x)=\frac{1}{\lambda(x)}\int_0^x\lambda(u)\,df(u). \tag{4.2}$$

Then

$$f(x)-f(0)=\int_0^x df(u)=\int_0^x\frac{1}{\lambda(u)}\lambda(u)\,df(u)=$$

$$=\int_0^x\frac{1}{\lambda(u)}\,d(\delta(u)\lambda(u))=\delta(x)+\int_0^x\delta(u)\frac{d\lambda(u)}{\lambda(u)},$$

and by (2.14),

$$f(x)=f(0)+\delta(x)+\int_0^x \delta(u)\frac{\rho'(u)}{\varphi(\rho(u))}\,du.$$

Let also

$$H(t,y)=\begin{cases}\dfrac{\varphi(\rho(x))}{|\rho(t)-\rho(x)|}\displaystyle\int_t^x \delta(u)\frac{\rho'(u)}{\varphi(\rho(u))}\,du, & t\neq x\\ 0, \quad t=x.\end{cases}$$

Then, by linearity of A,

$$|A(f,x)-f(x)A(1,x)|=|A(f(t)-f(x),x)|\leqslant$$
$$\leqslant|A(\delta(t),x)|+|\delta(x)|\,|A(1,x)|+\left|A\left(H(t,x)\frac{|\rho(t)-\rho(x)|}{\varphi(\rho(x))},x\right)\right|.$$

By (4.2), (2.13) and Theorem I, it follows that

$$|A(f,x)-f(x)A(1,x)|\leqslant o(1)+\left|A\left(H(t,x)\frac{|\rho(t)-\rho(x)|}{\varphi(\rho(x))},x\right)\right|,\quad(x\to\infty). \tag{4.3}$$

It will now be shown that $H(t,x)=o(1)$ $(x\to\infty)$, uniformly in $t\in\mathbf{R}^+$. Suppose first that $\rho(t)\leqslant\rho(x)/2$. Then

$$|H(t,x)|\leqslant 2\frac{\varphi(\rho(x))}{\rho(x)}\left|\int_t^x \delta(u)\frac{\rho'(u)}{\varphi(\rho(u))}\,du\right|.$$

Since $\rho(x)$ is monotone, let $\varepsilon(x)=|\delta(\rho^{-1}(x))|$. Then

$$\frac{\varphi(\rho(x))}{\rho(x)}\left|\int_0^t \delta(u)\frac{\rho'(u)}{\varphi(\rho(u)}\,du\right|\leqslant\frac{\varphi(\rho(x))}{\rho(x)}\int_0^{\rho(x)}\frac{\varepsilon(u)}{\varphi(u)}\,du=$$

$$<\frac{\varphi(\rho(x))\,\Phi(\rho(x))}{\rho(x)}\;\frac{1}{\Phi(\rho(x))}\int_0^{\rho(x)}\frac{\varepsilon(u)}{\varphi(u)}\,du=\eta_1(x). \tag{4.4}$$

Therefore

$$|H(t,x)|\leqslant 4\eta_1(x),\quad\text{if}\quad\rho(t)\leqslant\rho(x)/2.$$

But, the first factor of (4.4) is bounded in view of the last part of (2.8); and the second factor goes to zero by the definition of $\Phi(x)$, (2.15) and the fact that $\varepsilon(u)\to 0$ $(u\to\infty)$. Hence, $\eta_1(x)=o(1)$ $(x\to\infty)$.

Suppose now that $\rho(t)>\rho(x)/2$. Then

$$|H(t,x)|\leqslant\frac{\varphi(\rho(x))}{\varphi(\rho(x)/2)}\;\frac{1}{|\rho(t)-\rho(x)|}\left|\int_t^x|\delta(u)|\rho'(u)\,du\right|\leqslant$$

$$\leqslant\frac{\varphi(\rho(x))}{\varphi(\rho(x)/2)}\sup\{|\delta(u)|:u\geqslant\rho^{-1}(\rho(x)/2)\}=\eta_2(x). \tag{4.5}$$

The first factor of (4.5) is bounded, by (2.8), and the second goes to zero, by (4.2) and (2.13). Hence, $\eta_2(x)=o(1)$ $(x\to\infty)$. Thus, for any $t\geq 0$,

$$|H(t,x)|\leq 4\,\eta_1(x)+\eta_2(x)=o(1)\quad (x\to\infty). \tag{4.6}$$

Also, for every fixed $x>0$, and $t>x$,

$$|H(t,x)|\leq \frac{1}{\rho(t)-\rho(x)}\int_x^t |\delta(u)|\,\rho'(u)\,du=o(1)\quad (t\to\infty).$$

Thus, by (4.6) and definition (2.3), inequality (4.3) becomes

$$|A(f,x)-f(x)A(1,x)|\leq o(1)+o(1)\,W_A\left(\frac{\rho(t)-\rho(x)}{\varphi(\rho(x))},\,x\right)\quad (x\to\infty).$$

Result (4.1) now follows from hypothesis (2.12).

Proof of theorem 2. A slowly varying function $s(x)$ can be written as

$$s(x)=\exp\left\{\eta(x)+\int_0^x \frac{\varepsilon(t)}{t}\,dt\right\},$$

where $\eta(x)$, $\varepsilon(x)$ are measurable and bounded on $\mathbf{R}^+$, $\eta(x)\to c\in R$, $\varepsilon(x)\to 0$, $(x\to\infty)$, and $\varepsilon(x)=0$ in some neighborhood of 0. It is easy to deduce from this representation that, given any $\delta>0$, there exists M_δ such that

$$\frac{s(t)}{s(x)}\leq M_\delta x^\delta t^{-\delta},\quad \text{for } 0\leq t<x,\ x>1, \tag{5.1}$$

and

$$\frac{s(t)}{s(x)}\leq M_\delta x^{-\delta} t^\delta,\quad \text{for } 1<x<t. \tag{5.2}$$

In order to prove (3.4), in view of (3.3) and the linearity of A, it is sufficient to prove that

$$\left|A\left(\frac{f(t)-f(x)}{s(x)},\,x\right)\right|=o(1)\quad (x\to\infty). \tag{5.3}$$

Let $\alpha\in(0,1)$ and define

$$I_\alpha(x)=A\left(\frac{f(t)-f(x)}{s(x)}\chi_{[0,\alpha x)}(t),\,x\right)$$

and

$$J_\alpha(x)=A\left(\frac{f(t)-f(x)}{s(x)}\chi_{[\alpha x,\infty)}(t),\,x\right).$$

Then

$$\left|A\left(\frac{f(t)-f(x)}{s(x)},\,x\right)\right|\leq |I_\alpha(x)|+|J_\alpha(x)|. \tag{5.4}$$

Hence, in order to prove (5.3), it is sufficient to show that

$$\lim_{\alpha\to 0}\limsup_{x\to\infty}\,(|I_\alpha(x)|+|J_\alpha(x)|)=0.$$

To this end, we shall evaluate the expression $(f(t)-f(x))/s(x)$ for t in the two intervals $[0, x)$ and $[x, \infty)$. To simplify the computation, let us introduce the notation

$$\omega_\lambda(x)=\max_{x\leqslant t\leqslant \lambda x}\frac{|f(t)-f(t)|}{s(x)}. \tag{5.5}$$

Suppose first that $t<x$ and $x>1$. Let $\lambda>1$ be such that condition (3.5) be satisfied. Let σ be defined by $\lambda^{-\sigma}x=\max\{t, 1\}$, and let

$$N(t, x)=\begin{cases}\dfrac{f(t)-f(1)}{s(x)} & \text{if}\quad 0\leqslant t<1,\\ 0 \quad \text{if} \quad t\geqslant 1.\end{cases}$$

Then

$$\frac{f(t)-f(x)}{s(x)}=N(t, x)+\frac{f(\lambda^{-\sigma}x)-f(\lambda^{-[\sigma]}x)}{s(\lambda^{-\sigma}x)}\,\frac{s(\lambda^{-\sigma}x)}{s(x)}+$$
$$+\sum_{k=0}^{[\sigma]-1}\frac{f(\lambda^{-[\sigma]+k}x)-f(\lambda^{-[\sigma]+k+1}x)}{s(\lambda^{-[\sigma]+k}x)}\,\frac{s(\lambda^{-[\sigma]+k}x)}{s(x)},$$

where the last sum is zero if $0<\sigma<1$. Hence, by (5.5)

$$\frac{|f(t)-f(x)|}{s(x)}\leqslant|N(t, x)|+([\sigma]+1)\sup_{t\leqslant u\leqslant x}\omega_\lambda(u)\sup_{t\leqslant u\leqslant x}\frac{s(u)}{s(x)}. \tag{5.6}$$

Let η_0 be defined by the conditions of Theorem II, and let $\eta\in(0, \eta_0)$. Since, on $[0, 1]$, f is bounded and s is bounded away from 0, one can write, for $0\leqslant t<x$, and $x>1$.

$$|N(t, x)|\leqslant M_1 x^\eta. \tag{5.7}$$

On the other hand, since $\sigma=(\log x-\log(\max(t, 1)))\log^{-1}\lambda$, for $t<x$ and $x>1$ there is a constant M' such that

$$[\sigma]+1\leqslant M' x^{\eta-\delta}\mathbf{t}^{-(\eta-\delta)},$$

where $\delta\in(0, \eta)$. Hence, by (5.1), there exists a constant M'' such that

$$([\sigma]+1)\frac{s(t)}{s(x)}\leqslant M'' x^\eta \mathbf{t}^{-\eta}. \tag{5.8}$$

Therefore, by (5.6), (5.7) and (5.8), there is a constant M_2 such that, for $x>1$,

$$\frac{|f(t)-f(x)|}{s(x)}\leqslant M_2 x^\eta(1+\sup_{t\leqslant u\leqslant x}\omega_\lambda(u)),\quad 0\leqslant t\leqslant 1, \tag{5.9}$$

and

$$\frac{|f(t)-f(x)|}{s(x)}\leqslant M_2 x^\eta \mathbf{t}^{-\eta}\sup_{t\leqslant u\leqslant x}\omega_\lambda(u),\quad 1<t. \tag{5.10}$$

Suppose next that $t>x$. Let λ be as above and let $t=\lambda^\sigma x$, i.e. $\sigma = (\log t/x)/\log \lambda$. Then, by a chain of arguments similar to the above, using (5.2) instead of (5.1), one obtains, for $\eta \in (0, \eta_0)$,

(5.11) $$\frac{|f(t)-f(x)|}{s(x)} \leq M_3 x^{-\eta} t^{\eta} \sup_{x \leq u \leq t} \omega_\lambda(u), \text{ for } 1<x<t.$$

Note that

(5.12) $$\frac{f(t)-f(x)}{s(x)} = o(t^\eta) \quad (t \to \infty).$$

Indeed, it follows from the fact that (5.11) is also true if η is replaced by η', where $0<\eta'<\eta$.

Now, let $t<\alpha x$, and $\alpha x>1$. By (5.9), (5.10) and the fact that $\omega_\lambda(x)$ is bounded,

$$\frac{|f(t)-f(x)|}{s(x)} \leq M_4 x^\eta \mathbf{t}^{-\eta}.$$

Let $\gamma \in (0, \eta-\eta_0)$. Then, for all $t \geq 0$ we have

$$\frac{|f(t)-f(x)|}{s(x)} \chi_{[0,\alpha x)}(t) \leq M_4 (x^{-\gamma}/\mathbf{t}^{-\gamma}) x^{\eta+\gamma} \mathbf{t}^{-\eta-\gamma} \chi_{[0,\alpha x)}(t) \leq$$

$$\leq M_4 \alpha^\gamma x^{\eta+\gamma} \mathbf{t}^{-\eta-\gamma}.$$

It follows that

$$|I_\alpha(x)| \leq M_4 \alpha^\gamma x^{\eta+\gamma} W_A(\mathbf{t}^{-\eta-\gamma}, x).$$

Hence, by hypothesis (3.2)

(5.13) $$\limsup_{x \to \infty} |I_\alpha(x)| \leq M \alpha^\gamma.$$

Next, by (5.10) and (5.11), one has, for all $t \geq 0$,

$$\frac{|f(t)-f(x)|}{s(x)} \chi_{[\alpha x, \infty)}(t) \leq M_3 (x^\eta \mathbf{t}^{-\eta} + x^{-\eta} t^\eta) \sup_{\alpha x \leq u} \omega_\lambda(u).$$

Hence, by (5.12),

$$|J_\alpha(x)| \leq M_3 (x^\eta W_A(\mathbf{t}^{-\eta}, x) + x^{-\eta} W_A(t^\eta, x)) \sup_{\alpha x \leq u} \omega_\lambda(u).$$

It follows, by (3.2), (5.5) and (3.5), that

(5.14) $$\limsup_{x \to \infty} |J_\alpha(x)| = 0.$$

Finally, (5.3) follows from (5.4), (5.13) and (5.14), by taking α arbitrarily small.

REFERENCES

[1] L. V. Kantorovič, B. Z. Vulih and A. G. Pinsker, *Functional Analysis in partially ordered spaces* (in Russian) (Moscow, 1950).

[2] H. Nakano, *Modulared semi-ordered linear spaces* (Tokyo. 1950).

[3] B. Z. Vulih, *Introduction to the theory of partially ordered spaces*, (in Russian) (Moscow, 1961).

[4] R. Bojanic and M. Vuilleumier, *Asymptotic Properties of Linear Operators*, L'Ens. Math. **19** (1973), 283—308.

[5] G. H. Hardy, *Divergent Series* (Oxford, Clarendon Press, 1949).

[6] R. Raff, *Lineare Transformationen beschränkter integrierbarer Funktionen*, Math. Z. **41** (1936), 605—629.

[7] J. Karamata, *Sur les théorèmes inverses des procédès de sommabilité*, (Paris, Herman, 1937).

[8] G. E. Peterson, *Tauberian Theorems for Integrals* I, J. London Math. Soc. (2), **4** (1972), 493—498.

[9] J. Karamata, *Sur un mode de croissance régulière des fonctions*, Mathematica (Cluj) **4** (1930), 39—53.

[10] J. Karamata, *Sur un mode de croissance régulière, théorèmes fondamentaux*, Bull. Soc. Math. France **61** (1933), 55—62.

[11] W. Feller, *An introduction to probability theory and its application*, Voll. II, Second Edition (Wiley, New York, 1971).

[12] M. Vuilleumier, *Comportement asymptotique des transformations linéaires des suites*, Thèse, (Genève, 1966).

[13] M. Vuilleumier, *Sur le comportement asymptotique des transformations linéaires des suites*, Math. Z. **98** (1967), 126—139.

[14] H. Baumann, *Umkehrsätze für das asymptotische Verhalten linearer Folgentransformationen*, Math. Z. **98** (1967), 140—178.

[15] R. Bojanic and E. Senata, *Slowly Varying Functions and Asymptotic Relations*, J. Math. Anal. Appl., **34** (1971), 302—315.

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