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THEOREM ON DIFFERENTIAL EQUATIONS FOR MIKUSIŃSKI OPERATORS

Bogoljub Stanković

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The field $\mathcal{M}$ of Mikusiński operators is the extension of the integral domain $\mathcal{C}$ [6]. The relative poor topological structure of Mikusiński operators field $\mathcal{M}$ [1], [13] is the reason that it is difficult to investigate the differential equations in $\mathcal{M}$ especially nonlinear [4], [7], [8], [10], [11]. In this paper we prove a theorem for differential equations in $\mathcal{M}$ and apply it to a nonlinear differential equation and to a system of linear differential equations.

As the field $\mathcal{M}$ contains the integral operator l, the differential operator s and the translation operator, the differential equations in $\mathcal{M}$ cover some classes of partial differential equations, integral equations, difference equations and their combinations for numerical functions and are interesting for applications.

1. Let $\mathcal{J}\equiv[0, \infty)$ and $f\equiv\{f(t)\}$ be the representation of $f(t)\in\mathcal{C}_{\mathcal{J}}$ in $\mathcal{C}$. Let $\mathcal{M}(\lambda)$ be the vector space of mappings which maps the interval $\Omega\equiv[0, \Lambda]$ into $\mathcal{M}$ and $\mathcal{C}(\lambda)$ be the subspace of $\mathcal{M}(\lambda)$ of those elements which can be written in the form $f(\lambda)=\{f(\lambda, t)\}$, where $f(\lambda, t)\in\mathcal{C}_{\Omega\times\mathcal{J}}$. In $\mathcal{C}(\lambda)$ is defined an ordering relation $\leqslant$ $:f(\lambda)\leqslant_T g(\lambda) \Leftrightarrow f(\lambda, t)\leqslant g(\lambda, t)$, $(\lambda, t)\in\Omega\times[0, T]$; similar $f(\lambda)=\mathcal{O}_T[g(\lambda)]$. The absolute value is also introduced in $\mathcal{C}(\lambda): |f(\lambda)|=$ $=\{|f(\lambda, t)|\}$. An element $f(\lambda)\in\mathcal{C}(\lambda)$ is equal to zero if and only if $f(\lambda, t)=0$, $(\lambda, t)\in\Omega\times J$. Let ν_T be a saturated family of semi-norms in $\mathcal{C}(\lambda)$:

$$\nu_T[f(\lambda)]=\operatorname*{Max}_{(\lambda, t)\in\Omega\times[0, T]}|f(\lambda, t)|\equiv f_T \quad 0<T<\infty.$$

In $\prod\limits^{m}\mathcal{C}(\lambda)$ the family of semi-norms is:

$$N_T[\boldsymbol{x}(\lambda)]\equiv \operatorname*{Max}_{1\leqslant i\leqslant m} \nu_T[x_i(\lambda)]\equiv\boldsymbol{x}_T.$$

where $\boldsymbol{x}(\lambda)=\{x_1(\lambda), x_2(\lambda), \ldots, x_m(\lambda)\}$. In the same manner we have: $|\boldsymbol{x}(\lambda)|=$ $=\{|x_1(\lambda)|, \ldots, |x_m(\lambda)|\}$ and $\boldsymbol{x}(\lambda)\leqslant\boldsymbol{y}(\lambda) \Leftrightarrow x_i(\lambda)\leqslant y_i(\lambda)$, $i=1, 2, \ldots, m$.

Let $\boldsymbol{x}_0\in\prod\limits^{m}\mathcal{M}$ and the family (q_α), $\alpha\in\mathcal{G}$, be from $\mathcal{M}$. By definition $\mathcal{D}(\lambda)$ is a subset of $\prod\limits^{m}\mathcal{M}(\lambda)$ for which: 1. $x_0\in\mathcal{D}(\lambda)$; 2. For every $\alpha\in\mathcal{G}$ and

$a(\lambda)$, $b(\lambda)\in\mathcal{D}(\lambda)$, $q_\alpha[a(\lambda)-b(\lambda)]\in\prod^m\mathcal{C}(\lambda)$. In $\mathcal{D}(\lambda)$ we suppose a family of pseudodistances d_α to be defined [2], [5]: $d_\alpha[a(\lambda), b(\lambda)]=|q_\alpha[a(\lambda)-b(\lambda)]|$. In our case d_α, $\alpha\in\mathcal{J}$, maps $\mathcal{D}(\lambda)\times\mathcal{D}(\lambda)$ into $\prod^m\mathcal{C}(\lambda)$ and induces in $\mathcal{D}(\lambda)$ a convergence class: $a_n(\lambda)\in\mathcal{D}(\lambda)$ converges to $a(\lambda)\in\mathcal{D}(\lambda)$ if and only if for every $\alpha\in\mathcal{J}$, $d_\alpha[a_n(\lambda), a(\lambda)]\Rightarrow 0$ in $\prod^m\mathcal{C}(\lambda)$. The defined limit is unique. The definition of the derivative and the integral in $\mathcal{D}(\lambda)$ is this induced from $\prod^m\mathcal{M}(\lambda)$. The relation betwen the convergence class $ccl[\mathcal{D}(\lambda), \mathcal{M}(\lambda)]$ induced from $\mathcal{M}(\lambda)$ into $\mathcal{D}(\lambda)$ and that defined by the pseudodistances $ccl[\mathcal{D}(\lambda), d_\alpha]$ is: $ccl[\mathcal{D}(\lambda), d_\alpha]\subset ccl[\mathcal{D}(\lambda), \mathcal{M}(\lambda)]$, so that every sequence which converges in $[\mathcal{D}(\lambda), d_\alpha]$, converges in $\prod^m\mathcal{M}(\lambda)$ too.

Theorem. *Let $\mathcal{K}(\lambda)$ be a sequentially complete subspace of $\mathcal{D}(\lambda)$ and φ a mapping of $\mathcal{J}$ into $\mathcal{J}$. We suppose that for every $\lambda\in\Omega$, $\alpha\in\mathcal{J}$ and $x(\lambda)$, $y(\lambda)\in\mathcal{K}(\lambda)$:*

1. *f maps $\Omega\times\mathcal{K}(\lambda)$ into $\mathcal{D}(\lambda)$;*

2. *The mapping $R: R\,x(\lambda)=x_0+\int_0^\lambda f[u, x(u)]\,du$ maps $\mathcal{K}(\lambda)$ into $\mathcal{K}(\lambda)$.*

3. *There exists $r(\alpha)\in\mathcal{C}^+$ so that:*

$$r(\alpha)\, d_{\varphi^{k-1}(\alpha)}\{f[\lambda, x(\lambda)], 0\}\leqslant \nu(\alpha)\,\delta^{k-1} a_{x,\alpha}(\lambda)$$

where $\delta\in\mathcal{R}^+$, $\nu(\alpha)\in\mathcal{R}^+$, $a_{x,\alpha}(\lambda)\in\prod^m\mathcal{C}^+(\lambda)$.

4. *For every $x(\lambda)$, $y(\lambda)\in\mathcal{K}(\lambda)$ and $\alpha\in\mathcal{J}$ exists a sequence of matrices $\{B_i(\lambda)\}$ over $\mathcal{C}^+(\lambda)$ of a type $m\times m$ with the property:*

$$\prod_{n=1}^{k-1} B_n(u_n)=\{b^k_{i,j}(u_n)\},\quad 0\leqslant u_n\leqslant\Lambda,\quad b^k_{i,j}(u_n)\leqslant_T C\,r(\alpha)\,\Gamma^\beta(k+1)$$

where $\beta<1$, $C\in\mathcal{R}^+$, and so that:

$$d_{\varphi^i(\alpha)}\{f[\lambda, x(\lambda)], f[\lambda, y(\lambda)]\}\leqslant\frac{\mu(\alpha)}{\delta}B_{i+1}(\lambda)\, d_{\varphi^{i+1}(\alpha)}[x(\lambda), y(\lambda)],\ \mu(\alpha)\in\mathcal{R}^+.$$

The majorant of $b^k_{i,j}(\lambda)$ does not depend on $x(\lambda)$ and $y(\lambda)$ when they belong to the sequence $\{x_n(\lambda)\}$, $x_k(\lambda)=R\,x_{k-1}(\lambda)$.

Then there exists one and only one solution of the differential equation:

$$x'(\lambda)=f[\lambda, x(\lambda)],\quad x(0)=x_0\in\mathcal{K}(\lambda)\tag{1}$$

in the subspace $\mathcal{K}(\lambda)$ and this solution can be constructed by the sequence $\{x_n(\lambda)\}$.

Proof. Over the space $\mathcal{K}(\lambda)$ the differential equation (1) and the integral equation:

$$x(\lambda) = x_0 + \int_0^\lambda f[u, x(u)]\,du \tag{2}$$

are equivalent. We shall prove this.

Let us suppose that we have a solution $x(\lambda)$ in $\mathcal{K}(\lambda)$ of the equation (2). According to the definition of the integral in $\mathcal{M}(\lambda)$ there exists $q \in \mathcal{M}$ such that $q f(\lambda, x(\lambda)] = F(\lambda) \in \prod^{n} \mathcal{C}(\lambda)$ and

$$\int_0^\lambda f[u, x(u) du = \frac{1}{q}\left\{\int_0^\lambda F(u, t)\,du\right\}.$$

The equation (2) can be written now in the form:

$$q\,x(\lambda) = q\,x_0 + \left\{\int_0^\lambda F(u, t)\,du\right\}.$$

The function given by $\int_0^\lambda F(u, t)\,du$ has a continuous partial derivative in λ over $\mathcal{C}_{\Omega\times\mathcal{T}}$ and this derivative equals $F(\lambda, t)$. Then $x(\lambda) = x_0$ has a derivative in $\mathcal{M}(\lambda)$ too and for this derivative is:

$$[x(\lambda) - x_0]' = x'(\lambda) = \frac{1}{q}\{F(\lambda, t)\} = f[\lambda, x(\lambda)]$$

hence $x(\lambda)$ satisfies also the equation (1).

Let us suppose the opposite, i.e. that the equation (1) has a solution $x(\lambda) \in \mathcal{K}(\lambda)$. By the definition of the derivative in $\mathcal{M}(\lambda)$, there exists an element $p \in \mathcal{M}$ such that $p\,x(\lambda) = F(\lambda) \in \prod^{m} \mathcal{C}(\lambda)$ and $F(\lambda, t)$ has a continuous partial derivative in λ. To the equations (1) corresponds the equation

$$\{F'_\lambda(\lambda, t)\} = pf[\lambda, x(\lambda)] \in \mathcal{C}(\lambda).$$

After a formal integration in $\mathcal{C}(\lambda)$:

$$\{F(\lambda, t)\} - \{F(0, t)\} = \int_0^\lambda pf[u, x(u)]\,du$$

hence

$$x(\lambda) = x_0 + \int_0^\lambda f[u, x(u)]\,du$$

Let us construct now the sequence $x_k(\lambda) = R x_{k-1}(\lambda)$, $k = 1, 2, \ldots$ This sequence belongs to $\mathcal{K}(\lambda)$ because $x_0 \in \mathcal{K}(\lambda)$ too. We shall show that it is a Cauchy sequence:

$$d_\alpha[x_k(\lambda), x_{k-1}(\lambda)] = d_\alpha[R x_{k-1}(\lambda), R x_{k-2}(\lambda)] \leqslant$$

$$\leqslant \int_0^\lambda d_\alpha[f(u, x_{k-1}(u)), f(u, x_{k-2}(u))]\, du$$

$$\leqslant \frac{\mu(\alpha)}{\delta} \int_0^\lambda B_1(u)\, d_{\varphi(\alpha)}[x_{k-1}(u), x_{k-2}(u)]\, du$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\leqslant \frac{\mu^{k-1}(\alpha)}{\beta^{k-1}} \int_0^\lambda du_1 \ldots \int_0^{u_{k-2}} \prod_{n=1}^{k-1} B_n(u_n)\, d_{\varphi^{k-1}(\alpha)}[x_1(u_{k-1}), x_0]\, du_{k-1}$$

and

$$r(\alpha)\, d_{\varphi^{k-1}(\alpha)}[x_1(u_{k-1}), x_0] = r(\alpha)\, d_{\varphi^{k-1}(\alpha)}[R x_0, x_0]$$

$$\leqslant \int_0^{u_{k-1}} r(\alpha)\, d_{\varphi^{k-1}(\alpha)}\{f[u_k, x_0], 0\}\, du_k$$

$$\underset{T}{\leqslant} \nu(\alpha)\, \delta^{k-1} (a_{x_0, \alpha}(\lambda))_T \cdot u_{k-1}.$$

Now for every $\alpha \in \mathcal{G}$ and $T < \infty$ there exists $\varepsilon > 0$, such that

$$d_\alpha[x_k(\lambda), x_{k-1}(\lambda)] = O_T(z_k), \quad z_{k,i} = \frac{1}{(k!)^\varepsilon}, \qquad i = 1, \ldots, m.$$

After this inequality it is easy to show that the sequence $x_n(\lambda)$ is a Cauchy sequence. Let $x(\lambda)$ be its limit. We shall show that $x(\lambda)$ is the demanded solution of the equation (2), respectively equation (1).

The initial value $x(0) = x_0$ is satisfied because it is satisfied by every member of the sequence $x_n(\lambda)$. For every $\alpha \in \mathcal{G}$

$$d_\alpha[x(\lambda), R x(\lambda)] \leqslant d_\alpha[x(\lambda), x_k(\lambda)] + d_\alpha(x_k(\lambda), R x(\lambda)]$$

$$\leqslant d_\alpha[x(\lambda), x_k(\lambda)] + d_\alpha[R x_{k-1}(\lambda), R x(\lambda)]$$

$$\leqslant d_\alpha[x(\lambda), x_k(\lambda)] + \frac{\mu(\alpha)}{\delta} \int_0^\lambda B_1(u)\, d_{\varphi(\alpha)}[x_{k-1}(u), x(u)]\, du.$$

We know that the first and the second operation in $\mathcal{M}$ are continuous so the second part of this inequality tends to zero when $k \to \infty$. It follows that $x(\lambda) = R x(\lambda)$.

It remain only to prove that the found solution is unique in $\mathcal{K}(\lambda)$. Let us suppose that we have at least two solutions in $\mathcal{K}(\lambda): x(\lambda)$ and $y(\lambda)$, then:

$$d_\alpha[x(\lambda), y(\lambda)] = d_\alpha[R\,x(\lambda), R\,y(\lambda)]$$
$$\leqslant \int_0^\lambda d_\alpha\{f(u, x(u)], f[u, y(u)]\}\,du$$
$$\leqslant \frac{\mu(\alpha)}{\delta}\int_0^\lambda B_1(u)\,d_{\varphi(\alpha)}[x(u), y(u)]\,du$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\leqslant \frac{\mu^k(\alpha)}{\delta^k}\int_0^\lambda du_1\ldots\int_0^{u_{k-1}}\prod_{n=1}^k B_n(u_n)\,d_{\varphi^k(\alpha)}[x(u_k), y(u_k)]\,du_k$$
$$\leqslant \frac{\mu^k(\alpha)}{\delta^k}\int_0^\lambda du_1\ldots\int_0^{u_{k-1}}\prod_{n=1}^k B_n(u_n)\{d_{\varphi^k(\alpha)}[x(u_k), x_0]+$$
$$+d_{\varphi^k(\alpha)}[x_0, y(u_k)]\}\,du_k$$
$$=O_T(z_k),\quad z_{k,i}=\frac{2}{(k!)^\varepsilon},\quad i=1,\ldots,m,\quad \varepsilon>0,\quad T<\infty,$$

whence $x(\lambda)=y(\lambda)$.

2. We shall now apply our theorem to two special cases stressing the nature of the conditions suppoced in it. Both families of these equations have their own sense.

P r o p o s i t i o n 1. *The nonlinear differential equation*

$$y'(\lambda)=s^\beta a(\lambda)\,y^{m+1}(\lambda),\quad y(0)=l^{\frac{\beta}{m}}(\lambda),\quad m\geqslant 1,\quad \beta\geqslant 0 \tag{3}$$

has a unique solution in the set $y_0 l^{\frac{\beta}{m}}[I+C(\lambda)]$ *if* $y_0^m a(\lambda)\in\mathcal{C}(\lambda)$; I *is the unique element in* $\mathcal{M}$.

P r o o f. — We shall introduce the change of variables: $y(\lambda)=m^{\frac{1}{m}}y_0\,x(\lambda)$ and the equation (3) will obtain the form:

$$x'(\lambda)=s^\beta w(\lambda)\,x^{m+1}(\lambda),\quad x(0)=m^{-\frac{1}{m}}l^{\frac{\beta}{m}}\equiv x_0 \tag{4}$$

where $w(\lambda)=m\,a(\lambda)\,y_0^m\in\mathcal{C}(\lambda)$.

Before applying our theorem we shall prove three lemmas:

L e m m a 1. *Let* $S_p(x)$ *be the numerical sum:*

$$S_p(x)=m^{-\frac{1}{m}}\sum_{k=0}^{p}\frac{\Gamma\left(k+\frac{1}{m}\right)}{\Gamma(k+1)\,\Gamma\left(\frac{1}{m}\right)}x^k,\quad m\geqslant 1,\quad x\geqslant 0,\quad p\in\mathcal{N},$$

and $\hat{S}_p(x) = S_p(x) - m^{-\frac{1}{m}}$. *The following inequality is valued:*

$$l^{\frac{\beta}{m}+1} w_T \int_0^\lambda S_p^{m+1}(w_T ul)\, du \leqslant l^{\frac{\beta}{m}} \hat{S}_{(m+1)p+1}(w_T \lambda l), \tag{5}$$

where w_T *is a nonnegative number and* l *an integral operator in* $\mathcal{M}$.

Proof. Using the known relation:

$$\sum_{i=0}^{k} \frac{\Gamma(i+\alpha)}{\Gamma(i+1)\Gamma(\alpha)} \frac{\Gamma(k-i+\beta)}{\Gamma(k-i+1)\Gamma(\beta)} = \frac{\Gamma(k+\alpha+\beta)}{\Gamma(k+1)\Gamma(\alpha+\beta)} \tag{6}$$

we have

$$S_p(x)\, S_p(x) \leqslant m^{-\frac{2}{m}} \sum_{k=0}^{2p} \frac{\Gamma\left(k+\frac{2}{m}\right)}{\Gamma(k+1)\Gamma\left(\frac{2}{m}\right)} x^k.$$

With the same procedure we have:

$$S_p^{m+1}(x) \leqslant m^{-\frac{m+1}{m}} \sum_{k=0}^{(m+1)p} \frac{\Gamma\left(k+\frac{m+1}{m}\right)}{\Gamma(k+1)\Gamma\left(\frac{m+1}{m}\right)} x_k$$

$$\leqslant m^{-\frac{1}{m}} \sum_{k=1}^{(m+1)p+1} \frac{\Gamma\left(k+\frac{1}{m}\right)}{\Gamma(k+1)\Gamma\left(\frac{1}{m}\right)} kx^{k-1}.$$

Now

$$l^{\frac{\beta}{m}+1} w_T \int_0^\lambda S_p^{m+1}(w_T lu)\, du \leqslant l^{\frac{\beta}{m}} m^{-\frac{1}{m}} \int_0^\lambda \sum_{k=1}^{(m+1)p+1} \frac{\Gamma\left(k+\frac{1}{m}\right)}{\Gamma(k+1)\Gamma\left(\frac{1}{m}\right)} ku^{k-1}(w_T l)^k\, du$$

$$\leqslant l^{\frac{\beta}{m}} m^{-\frac{1}{m}} \sum_{k=1}^{(m+1)p+1} \frac{\Gamma\left(k+\frac{1}{m}\right)}{\Gamma(k+1)\Gamma\left(\frac{1}{m}\right)} (w_T \lambda l)^k.$$

Lemma 2. *Let* $w(\lambda) \in \mathcal{C}(\lambda)$ *and* $\mathcal{L}$ *be the set of those elements*

$$m^{-\frac{1}{m}} l^{\frac{\beta}{m}} [I + f(\lambda)], \quad f(\lambda) \in \mathcal{C}(\lambda), \quad m \geqslant 1, \quad \beta \geqslant 0,$$

for every of which there exists a number $p \geqslant 1$ *so that:* $m^{-\frac{1}{m}} f(\lambda) \underset{T}{\leqslant} \hat{S}_p(w_T \lambda l)$. *Then the mapping R:*

$$Rx(\lambda) = m^{-\frac{1}{m}} l^{\frac{\beta}{m}} + \int_0^\lambda s^\beta w(u)\, x^{m+1}(u)\, du$$

maps $\mathcal{L}$ *into* $\mathcal{L}$.

Proof. — Let $x(\lambda)$ be from $\mathcal{L}$:

$$Rx(\lambda)-m^{-\frac{1}{m}}l^{\frac{\beta}{m}}=\int_0^{\lambda} s^{\beta}w(u)\,x^{m+1}(u)\,du$$

$$\underset{T}{\leqslant}\int_0^{\lambda} w_T\,l\,l^{\frac{\beta}{m}}m^{-\frac{1}{m}}[I+f(u)]^{m+1}\,du$$

$$\underset{T}{\leqslant} l^{\frac{\beta}{m}}\int_0^{\lambda} w_T\,l\,S_p^{m+1}(w_T ul)\,du$$

$$\underset{T}{\leqslant} l^{\frac{\beta}{m}}\hat{S}_{(n+1)p+1}(w_T\lambda l).$$

We used here proposition of the lemma 1.

Lemma 3. *There exists* $F(\lambda)\in\mathcal{C}(\lambda)$ *such that* $\hat{S}_p(w_T\lambda l)\leqslant m^{-\frac{1}{m}}F(\lambda)$ *for all* $p\in\mathcal{N}$.

Proof. —

$$\hat{S}_p(w_T\lambda l)\leqslant m^{-\frac{1}{m}}\sum_{k=1}^{\infty}\frac{\Gamma\left(k+\frac{1}{m}\right)}{\Gamma(k+1)\Gamma\left(\frac{L}{m}\right)}(w_T\lambda l)^k.$$

This series converges in $\mathcal{C}(\lambda)$ and represents an element of $\mathcal{C}(\lambda)$.

Now we can apply our theorem to the equation (4).

The sequentially complete space $\mathcal{K}(\lambda)$ is the space of elements which has the form:

$$m^{-\frac{1}{m}}l^{\frac{\beta}{m}}[I+g(\lambda)],\quad g(\lambda)\in\mathcal{C}(\lambda).$$

The family (q_α) reduces to one element $s^{\frac{\beta}{m}}$ and $\delta=1$. The space $\mathcal{D}(\lambda)\equiv u(\lambda)\mathcal{K}(\lambda)$, $u(\lambda)\in(\mathcal{C}(\lambda)\cup I)$. We shall show that the conditions of our theorem are satisfied:

1. The function $f[\lambda, x(\lambda)]=s^{\beta}w(\lambda)\,x^{m+1}(\lambda)$ gives for $x(\lambda)\in\mathcal{K}(\lambda)$:

$$f[\lambda, x(\lambda)]=w(\lambda)\,l^{\frac{\beta}{m}}m^{-\frac{m+1}{m}}[I+g(\lambda)]^{m+1}\in\mathcal{D}(\lambda).$$

2. The mapping R maps $\mathcal{K}(\lambda)$ into $\mathcal{K}(\lambda)$: For $x(\lambda)\in\mathcal{K}(\lambda)$

$$Rx(\lambda)=m^{-\frac{1}{m}}l^{\frac{\beta}{m}}\left\{I+\int_0^{\lambda} w(u)\,m^{-1}[I+g(u)]^{m+1}\,du\right\}\in\mathcal{K}(\lambda).$$

3. $\left|s^{\frac{\beta}{m}}lf[\lambda, x(\lambda)]\right|=\left|m^{-\frac{m+1}{m}}lw(\lambda)[I+g(\lambda)]^{m+1}\right|$ whence

$$\nu(\alpha)=m^{-\frac{m+1}{m}},\quad r(\alpha)=l,\quad a_{x,\alpha}(\lambda)=\left|lw(\lambda)[I+g(\lambda)]^{m+1}\right|\in\mathcal{C}^{+}(\lambda).$$

4. For $x(\lambda)=m^{-\frac{1}{m}} l^{\frac{\beta}{m}}[I+g(\lambda)]$ and $y(\lambda)=m^{-\frac{1}{m}} l^{\frac{\beta}{m}}[I+h(\lambda)]$ the sequence of matrices $\{B_i(\lambda)\}$ is a stationary sequence of elements of $\mathcal{C}(\lambda)$:

$$B(\lambda)=m^{-1}\left| w(\lambda) \sum_{k=0}^{m} [I+g(\lambda)]^k [I+h(\lambda)]^{m-k} \right|$$

and

$$B^k(\lambda) \underset{T}{\leqslant} \frac{B_T^k T^{k-1}}{\Gamma(k-1)}.$$

Using lemmas 2 and 3 we have for $x(\lambda)$ and $y(\lambda)$, when they belong to the constructed sequence $x_n(\lambda)$:

$$B(\lambda) \underset{T}{\leqslant} m^{-1}(m+1) \left| w(\lambda)[I+F(\lambda)]^m \right|.$$

So we have all the suppositions of our theorem satisfied and the proposition 1 is proved.

Before we apply our theorem to a system of linear differential equations we shall give properties of some sets and functions we need.

Finite set of real numbers. Let us consider the set of m^2, $m \leqslant 2$, non-negative numbers $\beta_{i,j}$, $1 \leqslant i,\ j \leqslant m$. Let $(i_0, i_1, \ldots, i_\alpha)$ be a subset of first m integers such that $i_\alpha = i_0$; $i_k \neq i_j$, $k \neq j \neq \alpha$. We denote by $\sigma_\alpha(\beta_{i,j})$ the sum

$$\sigma_\alpha(\beta_{i,j}) = \sum_{k=0}^{\alpha-1} \beta_{i_k, i_{k+1}}.$$

The number of such sums is finite. Let ζ be the maximum value of the quotients $\sigma_\alpha(\beta_{i,j})/\alpha$ for all sums $\sigma_\alpha(\beta_{i,j})$. Every sum $\sum_{p=0}^{k-1} \beta_{i_p, i_{p+1}}$, $k>m$ may be expanded into a finite sum of σ_α—sums and a remainder P whose number of elements in the sum is less than m [3], and we have:

$$\sum_{p=0}^{k-1} \beta_{i_p, i_{p+1}} = \sum_{i=1}^{r} \sigma_{\alpha_i} + P \leqslant \zeta \sum_{i=1}^{r} \alpha_i + P \leqslant \zeta k + \gamma,$$

where γ is a constant independent of k.

The function $F_{p,k}(t)$. We shall use a special function:

$$F_{p,k}(t) = \begin{cases} t^{-p-1}\Phi\left(-p,\ -\sigma;\ -\frac{1}{2^k} t^{-\sigma}\right), & t>0 \\ 0, & t=0 \end{cases} \quad \begin{matrix} k \text{ integer}, \\ p \in \mathcal{R} \\ 0<\sigma<1, \end{matrix}$$

where Φ is the known function of E. M. Wright [14]. The properties of this functions, we need, are [12]:

1. $F_{p,k} \in \mathcal{C}$; 2. $F_{o,k} \geqslant 0$; 3. $s^\beta F_{p,k} = F_{p+\beta,k}$; 4. $F_{p,k+1} F_{q,k+1} = F_{p+q,k}$;

5. $$\prod_{i=1}^{k-1} F_{0,r+i} = \left\{ t^{-1}\Phi\left(0,\ -\sigma;\ -\sum_{i=1}^{k-1} \frac{1}{2^{r+i}} t^{-\sigma}\right)\right\};$$

6. $|F_{p,k}(t)| \leqslant 2^{k\left(\frac{p+1}{\sigma}\right)} N^p \Gamma\left(\frac{p+1}{\sigma}\right)$, N depends only on σ,

The space $\mathcal{C}^*(\lambda)$. Let us consider in $\mathcal{M}(\lambda)$ such elements as for every k integer have a representative in the equivalence class of the form:

$$\frac{\omega_k(\lambda)}{F_{0,k}}, \quad \omega_k(\lambda) \in \mathcal{C}(\lambda).$$

The subset of such elements we note by $\mathcal{C}^*(\lambda)$. It is a vector space and is not empty; we know that $\mathcal{C}(\lambda) \subset \mathcal{C}^*(\lambda)$ and $s^\beta \mathcal{C}(\lambda) \subset \mathcal{C}^*(\lambda)$, $\beta \subset \mathcal{R}$. For every k integer $F_{0,k}\mathcal{C}^*(\lambda) \subset \mathcal{C}(\lambda)$ and we can define a family of pseudodistances d_k in $\mathcal{C}^*(\lambda)$: $d_k[a(\lambda), b(\lambda)] = |F_{0,k}[a(\lambda) - b(\lambda)]|$. Our space $\mathcal{C}^*(\lambda)$ is sequentially complete [4]. We shall prove it.

Let $\eta_n(\lambda)$ be a Cauchy sequence of $\mathcal{C}^*(\lambda)$, i.e. for every k integer:

$$\nu_T[F_{0,k}\eta_n(\lambda) - F_{0,k}\eta_m(\lambda)] \to 0, \quad n, m \to \infty.$$

It follows that $F_{0,k}\eta_n(\lambda) = y_{n,k}(\lambda)$ is a Cauchy sequence in $\mathcal{C}(\lambda)$ for every fixed k. As $\mathcal{C}(\lambda)$ is complete, there exists $\omega_k(\lambda) \in \mathcal{C}(\lambda)$ which is the limit of this sequence for every k.

Let us consider the element $\eta(\lambda) = \dfrac{\omega_k(\lambda)}{F_{0,k}}$. We shall show that this element belongs to the set $\mathcal{C}^*(\lambda)$, i.e. $\omega_k(\lambda) F_{0,p} = \omega_p(\lambda) F_{0,k}$:

$$\nu_T[F_{0,p}\omega_k(\lambda) - F_{0,k}\omega_p(\lambda)] \leqslant$$
$$\leqslant \nu_T[F_{0,p}\omega_k(\lambda) - F_{0,p}F_{0,k}\eta_n(\lambda)] + \nu_T[F_{0,p}F_{0,k}\eta_n(\lambda) - F_{0,k}\omega_p(\lambda)]$$
$$\leqslant T\nu_T(F_{0,p})\nu_T(\omega_k(\lambda) - y_{n,k}(\lambda)) + T\nu_T(F_{0,k})\nu_T(y_{n,p}(\lambda) - \omega_p(\lambda)).$$

The second part of this inequality tends to zero when $n \to \infty$, for every k, p integers and $T < \infty$, and the proof is finished.

In the product $\prod^k \mathcal{C}^*(\lambda)$ we can bring over the structure of the vector space and the convergence class induced by the family d_k as it becomes customary, applying to coordinates.

Now we can prove the following proposition for a system of linear differential equations:

P r o p o s i t i o n 2. *Let us suppose:*

1. $a_{i,j}(\lambda) = s^{\beta_{i,j}}\omega_{i,j}(\lambda)$, $\omega_{i,j}(\lambda) \in \mathcal{C}(\lambda)$, $1 \leqslant i, j \leqslant m$;
2. $b_i(\lambda) \in \mathcal{C}^*(\lambda)$, $i = 1, \ldots, m$;
3. $\zeta = \operatorname{Max} \dfrac{\sigma_\alpha(\beta_{i,j})}{\alpha} < 2$.

Then the system:

$$x_i'(\lambda) = \sum_{j=1}^{m} a_{i,j}(\lambda) x_j(\lambda) + b_i(\lambda), \quad i = 1, 2, \ldots, m \tag{7}$$

with the initial conditions $x_i(0) = x_{i,o} \in \mathcal{C}^*(\lambda)$, $i = 1, \ldots, m$, *has a unique solution in* $\prod^m \mathcal{C}^*(\lambda)$.

Proof. The system (7) can be written in the vector form:

$$x'(\lambda) = A(\lambda)\, x(\lambda) + b(\lambda) \tag{8}$$

$A(\lambda)$ is the matrix $\{a_{i,j}(\lambda)\}$, $x(\lambda) = \{x_1(\lambda), \ldots, x_m(\lambda)\}$, $b(\lambda) = \{b_1(\lambda), \ldots, b_m(\lambda)\}$ and $x_0 = \{x_{1,0}, \ldots, x_{m,0}\}$.

We shall show that the conditions of our theorem are satisfied.

Let $\mathcal{K}(\lambda)$ be in this case $\prod^m \mathcal{C}^*(\lambda)$ and the family (q_α) be $(F_{0,k})$, k integer; the mapping φ: $\varphi(k) = k+1$.

1. $f[\lambda, x(\lambda)] = A(\lambda)\, x(\lambda) + b(\lambda)$. From the property 4 of the function $F_{p,k}$ follows:

$$F_{0,k} A(\lambda)\, x(\lambda) + F_{0,k}\, b(\lambda) = F_{0,k+1} A(\lambda)\, F_{0,k+1}\, x(\lambda) + F_{0,k}\, b(\lambda) \in \prod^m \mathcal{C}(\lambda).$$

2. The mapping R is: $R\,x(\lambda) = x_0 + \int_0^\lambda A(u)\, x(u)\, du + \int_0^\lambda b(u)\, du$. We know that x_0 and $b(\lambda)$ belong to $\prod^m \mathcal{C}^*(\lambda)$. It remains to show that

$$\int_0^\lambda A(u)\, x(u)\, du \in \prod^m \mathcal{C}^*(\lambda).$$

For this it is enough to see that:

$$\int_0^\lambda \sum_{j=1}^m s^{\beta_{i,j}} \omega_{i,j}(u) \frac{w_{j,k}(u)}{F_{0,k}}\, du \in \mathcal{C}^*(\lambda), \quad j = 1, \ldots, m, \quad x_j = \frac{w_{j,k}}{F_{0,k}}.$$

Let β be $\beta = \max_{1 \le i,j \le m} \beta_{i,j}$. Using the properties of the function $F_{0,k}$ we have:

$$\int_0^\lambda \sum_{j=1}^m s^{\beta_{i,j}} \omega_{i,j}(u) \frac{w_{j,k}(u)}{F_{0,k}}\, du = \int_0^\lambda \sum_{j=1}^m s^{-(\beta-\beta_{i,j})} \omega_{i,j}(u) \frac{w_{j,k}(u)}{F_{-\beta,k}}\, du$$

$$= \frac{I}{F_{0,k-1}} \int_0^\lambda \sum_{j=1}^m s^{-(\beta-\beta_{i,j})}\, \omega_{i,j}(u)\, w_{j,k}(u)\, F_{\beta,k}\, du.$$

The function given by the integral belongs to $\mathcal{C}(\lambda)$.

3. Let $\beta = \max_{1 \le i,j \le m} \beta_{i,j}$, $r(\alpha) = F_{0,\alpha}$, α integer, then

$$|F_{0,\alpha} F_{0,\alpha+k-1} f[\lambda, x(\lambda)]| = |F_{0,\alpha+k-1} A(\lambda)\, w_\alpha(\lambda) + F_{0,\alpha+k-1}\, b_\alpha(\lambda)|$$
$$= |F_{\beta,\alpha+k-1}|\, |l^\beta A(\lambda)\, w_\alpha(\lambda) + l^\beta b_\alpha(\lambda) \le$$
$$\le 2^{\frac{\beta+1}{\sigma}(k+\alpha-1)} N^\beta\, \Gamma\!\left(\frac{\beta+1}{\sigma}\right) |l^\beta A(\lambda)\, w_\alpha(\lambda) + l^\beta b_\alpha(\lambda)|$$

whence

$$\delta = 2^{\frac{\beta+1}{\sigma}}, \quad \nu(\alpha) = 2^{\frac{\beta+1}{\sigma}\alpha} N^\beta\, \Gamma\!\left(\frac{\beta+1}{\sigma}\right)$$

and

$$a_{x,\alpha}(\lambda) = |\, l^{\beta} A(\lambda)\, w_{\alpha}(\lambda) + l^{\beta} b_{\alpha}(\lambda)|; \quad w_{\alpha}(\lambda) = F_{0,\alpha}\, x(\lambda), \quad b_{\alpha}(\lambda) = F_{0,\alpha}\, b(\lambda).$$

4. $$|F_{0,\alpha+i}[A(\lambda)\, x(\lambda) - A(\lambda)\, y(\lambda)]| \leqslant \{|F_{0,\alpha+i+1}\, a_{i,j}(\lambda)|\} \times$$
$$\times |F_{0,\alpha+i+1}[x(\lambda) - y(\lambda)]|.$$

The sequence of matrices $\{B_k(\lambda)\}$ can be now: $B_k(\lambda) = \{|F_{0,\alpha+k}\, s^{\beta_{i,j}}\, \omega_{i,j}(\lambda)|\}$ and $\mu(\alpha) = 2^{\frac{\beta+1}{\sigma}}$. We see that $B_k(\lambda)$ does not depend on $x(\lambda)$ or $y(\lambda) \in \mathcal{K}(\lambda)$ and

$$b_{i,j}^{k} = \left| F_{0,\alpha} \prod_{p=1}^{k-1} F_{0,\alpha+p} \sum_{p_{k-2}=1}^{m} \cdots \sum_{p_1=1}^{m} a_{i,p_{k-2}}\, a_{p_{k-2},p_{k-1}} \cdots a_{p_1,j} \right|.$$

We shall fix the number σ in the function $F_{p,k}$ so that $\zeta/\sigma < 2$. Using the properties 3, 5 and 6 for the function $F_{p,k}$ and the inequality

$$\omega^k(\lambda) \leqslant_{T} \omega_T{}^k \frac{T^k}{\Gamma(k-1)}$$

we have for $b_{i,j}^k$: $b_{i,j}^k = O_T[F_{0,\alpha}\Gamma^{\beta}(k)]$, $\beta < 1$.

3. As an illustration of our proposition 2. let us consider the diffusion partial differential equation:

$$\frac{\partial u(\lambda, t)}{\partial t} = A(\lambda) \frac{\partial^2 u(\lambda, t)}{\partial \lambda^2} + B(\lambda) \frac{\partial u(\lambda, t)}{\partial \lambda}. \tag{9}$$

We know that $A(\lambda) > 0$ for $0 \leqslant \lambda \leqslant \Lambda$. To this equation corresponds in $\mathcal{M}(\lambda)$:

$$\frac{d^2 u(\lambda)}{d\lambda^2} + s \frac{B(\lambda)}{A(\lambda)}\, l \frac{du(\lambda)}{d\lambda} - s^2 \frac{I}{A(\lambda)}\, l u(\lambda) = -\frac{I}{A(\lambda)}\, u_0(\lambda) \tag{10}$$

where $u_0(\lambda) = \lim_{t \to 0+} u(\lambda, t)$. The equation (10) is equivalent to the system:

$$\begin{aligned} u_1'(\lambda) &= s l u_2(\lambda) \\ u_2'(\lambda) &= s^2 \frac{I}{A(\lambda)}\, l u_1(\lambda) - s \frac{B(\lambda)}{A(\lambda)}\, l u_2(\lambda) - \frac{I}{A(\lambda)}\, u_0(\lambda) \end{aligned} \tag{11}$$

in which $\beta_{1,1} = -\infty$, $\beta_{1,2} = 1$, $\beta_{2,1} = 2$, $\beta_{2,2} = 1$; whence $\zeta = \frac{3}{2} < 2$. If we suppose that $A(\lambda)$ and $B(\lambda) \in \mathcal{C}_{\Omega}$, the proposition 2 asserts the existence of the unique solution with the initial condition: $u(0, t)$ and $u'(0, t)$ from $\mathcal{C}^*(\lambda)$. The construction of the solution is given by the sequence $\{x_n(\lambda)\}$.

Let us remarque that the derivatives and limits are in the sense of operators and they have not to exist in the classical sense.

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