PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 18 (32), 1975 p. 173—179

AN EXTENSION OF RICE'S RESULT ON AN INTEGRAL EQUATION

Manilal Shah

(Received September 28, 1973)

In the present work, the autor has established an integral which involves Fox's H-function of two variables. It is further shown how such result is useful to evaluate the contour integral by an application of a theorem on Mellin inversion transform. Later on, this contour integral equation is utilized to investigate a more general formula of the nature of an integral equation which, in turn, yields an extension of a well-known result due to Rice. Many interesting results are also recorded on appropriately specializing the parameters.

1 Introduction: Fox's H-function of Two Variables

A generalization of Fox's H-function [4, p. 408] and Meijer's G-function of two variables [6, p. 27, (2)] has been recently introduced by Munot and Kalla [5, p. 68, (2)]:

$$\tag{1.1} H\left[\begin{matrix}\left[\begin{smallmatrix} m_1, & 0 \\ p_1-m_1, & q_1\end{smallmatrix}\right] \\ \left(\begin{smallmatrix} m_2, & n_2 \\ p_2-m_2, & q_2-n_2\end{smallmatrix}\right) \\ \left(\begin{smallmatrix} m_3, & n_3, \\ p_3-m_3, & q_3-n_3\end{smallmatrix}\right)\end{matrix}\middle|\begin{matrix}(a_{p_1}, A_{p_1}; (b_{q_1}, B_{q_1}) \\ (c_{p_2}, C_{p_2}); (d_{q_2}, D_{q_2}) \\ (e_{p_3}, E_{p_3}); (f_{q_3}, F_{q_3})\end{matrix}\middle|\begin{matrix} x \\ \\ y\end{matrix}\right]$$

$$=\frac{1}{(2\pi i)^2}\int_{L_1}\int_{L_1} F(\xi+\eta)\,\theta(\xi,\eta)\,x^{\xi}y^{\eta}\,d\xi\,d\eta,$$

where

$$F(\xi+\eta)=\frac{\prod_{j=1}^{m_1}\Gamma(a_j+A_j\xi+A_j\eta)}{\prod_{j=m_1+1}^{p_1}\Gamma(1-a_j-A_j\xi-A_j\eta)\prod_{j=1}^{q_1}\Gamma(b_j+B_1\xi+B_j\eta)},$$

$$\theta(\xi,\eta)=\frac{\prod_{j=1}^{m_2}\Gamma(1-c_j+C_j\xi)\prod_{j=1}^{n_2}\Gamma(d_j-D_j\xi)\prod_{j=1}^{m_3}\Gamma(1-e_j+E_j\eta)\prod_{j=1}^{n_3}\Gamma(f_j-F_j\eta)}{\prod_{j=m_2+1}^{p_2}\Gamma(c_j-C_j\xi)\prod_{j=n_2+1}^{q_2}\Gamma(1-d_j+D_j\xi)\prod_{j=m_3+1}^{p_3}(e_j-E_j\eta)\prod_{j=n_3+1}^{q_3}\Gamma(1-f_j+F_j\eta)}.$$

The integral (1.1) converges under the conditions defined in [5]:

$$
(1.2)\quad \begin{cases}
\lambda_1 \equiv \sum_1^{p_1} A_j + \sum_1^{p_2} C_j - \sum_1^{q_1} B_j - \sum_1^{q_2} D_j \leqslant 0;\ \mu_1 \equiv \sum_1^{p_1} A_j + \sum_1^{p_3} E_j - \sum_1^{q_1} B_j - \sum_1^{q_3} F_j \leqslant 0, \\
\lambda_2 = \sum_1^{m_1} A_j - \sum_{m_1+1}^{p_1} A_j - \sum_1^{q_1} B_j + \sum_1^{m_2} C_j - \sum_{m_2+1}^{p_2} C_j + \sum_1^{n_2} D_j - \\
- \sum_{n_2+1}^{q_2} D_j > 0,\ |\arg x| < \frac{1}{2}\lambda_2 \pi, \\
\mu_2 = \sum_1^{m_1} A_j - \sum_{m_1+1}^{p_1} A_j - \sum_1^{q_1} B_j + \sum_1^{m_3} E_j - \sum_{m_3+1}^{p_3} E_j + \sum_1^{n_3} F_j - \\
- \sum_{n_3+1}^{q_3} F_j > 0,\ |\arg y| < \frac{1}{2}\mu_2 \pi.
\end{cases}
$$

The paper is divided into three main parts. In **3** an integral associated with Fox's H-function of two variables is obtained. In **4** the importance of such result which lies in the evaluation of the contour integral by virtue of the Mellin and the inverse Mellin transform techniques is illustrated. In **5** a general fromula on an integral equation presenting Fox's H-function in two arguments is deduced. In the end, several interesting cases including Rice's result are derived.

2. Preliminaries

For the sake of brevity and printing convenience, we require the notations:

(i) the L. H. S. of (1.1) is denoted by $H\begin{bmatrix} x \\ y \end{bmatrix}$;

(ii) (a_p, A_p) stands for the set of parameters $(a_1, A_1), (a_2, A_2), \ldots, (a_p, A_p)$;

(iii) $|a|_m$ or a_m is taken to represent the sequence of m-parameter $a_1, a_2, \ldots, a_j, \ldots, a_m$,

(iv) $(a)_m = \dfrac{\Gamma(a+m)}{\Gamma(a)} = a(a+1)\cdots(a+a-1;\ m=1, 2, 3, \ldots$

(v) $\Delta(m, n) = \dfrac{n}{m}, \dfrac{n+1}{m}, \ldots, \dfrac{n+m-1}{m}$; m is a positive integer

(vi) $f = f\begin{pmatrix} m, n \\ p, q \end{pmatrix} = \begin{cases} \begin{bmatrix} m_1+h, & 0 \\ p_1-m_1, & q_1+h \end{bmatrix} \\ \begin{pmatrix} m_2, & n_2 \\ p_2-m_2, & q_2-n_2 \end{pmatrix} \\ \begin{pmatrix} m_3, & n_3 \\ p_3-m_3, & q_3-n_3 \end{pmatrix}, \end{cases}$

(vii) $g = \begin{cases} (c_{p_2}, C_{p_2});\ (d_{q_2};\ D_{q_2}) \\ (e_{p_3}, E_{p_3});\ (f_{q_3}, F_{q_3}). \end{cases}$

The following results will be used in the sequel:

(a) Integral [2, p. 9, (2)]:

$$\int_0^\infty \frac{x^{p-1}}{(1+x)^{p+q}}\,dx = \frac{\Gamma(p)\,\Gamma(q)}{\Gamma(p+q)}, \quad Re(p)>0, \; Re(q)>0, \tag{2.1}$$

(b) The Gamma-product formula [2, p. 4, (11)]:

$$\Gamma(mz) = (2\pi)^{\frac{1}{2}-\frac{1}{2}m}\, m^{mz-\frac{1}{2}} \prod_{k=0}^{m-1} \Gamma\left(z+\frac{k}{m}\right). \tag{2.2}$$

(c) Integral Transforms.

The Mellin transform [3, p. 305]:

$$g(s) = \int_0^\infty x^{s-1} f(x)\,dx = M_s\{f(x)\} \text{ or } M\{f(x);\, s\} \tag{2.3}$$

and its inversion formula [3, p. 307, (1)];

$$f(x) = M_x^{-1}\{g(s)\} = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} x^{-s} g(s)\,ds. \tag{2.4}$$

(d) Kampé de Fériet function

The abbreviated form of Kampé de Fériet's function is

$$F_{\nu,\rho}^{\lambda,\mu}\left[\begin{array}{c|c|c} |\,\alpha\,|_\lambda : |\,\beta,\ \beta'\,|_\mu & x \\ |\,\gamma\,|_\nu : |\,\delta,\ \delta'\,|_\rho & y \end{array}\right] \tag{2.5}$$

which reduces to a generalized hypergeometric function of one variable

$${}_{m+1}F_m\left[\begin{matrix}\alpha_m,\ \beta+\beta' \\ \gamma_m\end{matrix};\ x\right] \tag{2.6}$$

by giving particular values to the parameters.

3. The General Integral

In this section we establish an integral involving Fox's H-function of two variables which is required in the development of the present work.

An Integral Formula: *If h is a positive integer >0, $0<z<\infty$, then*

$$\int_0^\infty \frac{z^{l+s-1}}{(1+z)^{k+s}}\, H\left[\begin{matrix} x\left(\frac{z}{1+z}\right)^h \\ y\left(\frac{z}{1+z}\right)^h \end{matrix}\right] dz \tag{3.1}$$

$$= \frac{\Gamma(k-l)}{k^{k-l}}\, H\left[\bar{f}\,\Big|^{\Delta(h,\, l+s),\ (a_{p_1},\, A_{p_1})};\ g^{(b_{q_1},\, B_{q_1}),\ \Delta(h,\, k+s)}\Big|\begin{matrix}x\\y\end{matrix}\right]$$

where, for convergence,

$$p_1 \geqslant m_1 \geqslant 0,\ q_1 \geqslant 0,\ p_2 \geqslant m_2 \geqslant 0,\ q_2 \geqslant n_2 \geqslant 0,\ p_3 \geqslant m_3 \geqslant 0,$$

$$q_3 \geqslant n_3 \geqslant 0,\ Re(k-l)>0,\ Re\left(l+s+h\frac{d_i}{D_i}+h\frac{f_j}{F_j}\right)>0$$

$$(1 \leqslant i \leqslant n_2;\ 1 \leqslant j \leqslant n_3),\ \lambda_1 \leqslant 0,\ \mu_1 \leqslant 0,\ \lambda_2 > 0 \text{ and } \mu_2 > 0.$$

Proof: On the left of (3.1) replace the H-function of two variables by (1.1). Then inverting order of integration we find that it becomes

$$(3.2) \qquad \frac{1}{(2\pi i)^2}\int_{L_1}\int_{L_2} F(\xi+\eta)\,\theta(\xi,\eta)\,x^{\xi}y^{\eta}\left\{\int_0^{\infty}\frac{z^{l+s+h\xi+h\eta-1}\,dz}{(1+z)^{k+s+h\xi+h\eta}}\right\}d\xi\,d\eta.$$

To justify, the change in order of integration, we observe that

(i) the integral:

$$\int_0^{\infty}\frac{z^{l+s+h\xi+h\eta-1}}{(1+z)^{k+s+h\xi+h\eta}}\,dz$$

is absolutely convergent if $\mathrm{Re}(l+s+h\xi+h\eta)>0$, $\mathrm{Re}(k-l)>0$ and h is a non-negative integer >0;

(ii) the double contour integral converges absolutely under the conditions referred to earlier in (1.2) and convergence of the repeated integral follows from that of integral (3.1); and

(iii) Fox's H-function of two variables is a continuous function of x and y for all values of $x \geqslant x_0 > 0$ and $y \geqslant y_0 > 0$.

Hence the change of order of integration is permissible by Bromwich [1, p. 504].

Now, applying (2.1) and (2.2) in (3.2) to arrive at

$$(3.3) \qquad \frac{\Gamma(k-l)}{h^{k-l}}\frac{1}{(2\pi i)^2}\int_{L_1}\int_{L_2} F(\xi+\eta)\,\theta(\xi,\eta)\prod_{i=0}^{h-1}\left\{\frac{\Gamma\left(\frac{l+s+i}{h}+\xi+\eta\right)}{\Gamma\left(\frac{k+s+i}{h}+\xi+\eta\right)}\right\}x^{\xi}y^{\eta}\,d\zeta\,d\eta$$

where the contour L_1 in the ξ-plane is a straight line along the imaginary axis extending from $\rho-i\infty$ to $\rho+i\infty$ with indentation, if necessary to ensure that the poles of $\Gamma(d_j-D_j\xi)$ $(1 \leqslant j \leqslant n_2)$ lie to the right of it and the poles of $\Gamma(1-c_j+C_j\xi)$ $(1 \leqslant j \leqslant m_2)$,

$$\Gamma(a_j+A_j\zeta+A_j\eta)\ (1 \leqslant j \leqslant m_1) \text{ and } \Gamma\left(\frac{l+s+i}{h}+\xi+\eta\right)\{0 \leqslant i \leqslant (h-1)\}$$

lie to the left of the contour.

Similarly the contour L_2 in the η-plane runs from $\sigma - i\infty$ to $\sigma + i\infty$ with loops, if necessary to ensure that the poles due to $\Gamma(f - F_j\eta)$ $(1 \leqslant j \leqslant n_3)$ are to the right and the poles due to $\Gamma(1 - e_j + E_j\eta)$ $(1 \leqslant j \leqslant m_3)$, $\Gamma(a_j + A_j\xi + A_j\eta)$ $(1 \leqslant j \leqslant m_1)$ and $\Gamma\left(\frac{l+s+i}{h} + \xi + \eta\right)\{0 \leqslant 1 \leqslant (h-1)\}$ are left to the contour.

Finally, the application of (1.1) in (3.3) leads to (3.1).

4. The Contour Integral

Here we evaluate the contour integral equation for the H-functions of two variables with the aid of the theorems on integral transforms.

Next using (2.3) an (2.4) in (3.1) which readily gives

(4.1)
$$\frac{z^s}{(1+z)^{k+s}} H \begin{bmatrix} x\left(\frac{z}{1+z}\right)^h \\ y\left(\frac{z}{1+z}\right)^h \end{bmatrix}$$

$$= \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \frac{\Gamma(k-l)}{h^{k-l}} H\left[f\left|{}^{\Delta(h,\, l+s),\ (a_{p_1},\, A_{p_1})}; \ g^{(b_{q_1},\, B_{q_1}),\ \Delta(h,\, k+s)}\right. z^{-l}\, dl \Big|{}^x_y\right]$$

valid for $h>0$, $\operatorname{Re}(k-l)>0$, $p_1 \geqslant m_1 \geqslant 0$, $q_1 \geqslant 0$, $p_2 \geqslant m_2 \geqslant 0$, $q_2 \geqslant n_2 \geqslant 0$, $p_3 \geqslant m_3 \geqslant 0$. $q_3 \geqslant n_3 \geqslant 0$, $\operatorname{Re}\left(l+s+h\frac{d_i}{D_i}+h\frac{f_i}{F_i}\right)>0$, $(1 \leqslant i \leqslant n_2;\ 1 \leqslant j \leqslant n_3)$, $\lambda_1 \leqslant 0$, $\mu_1 \leqslant 0$, $\lambda_2 > 0$ and $\mu_2 > 0$

5. An integral Equation

In this section we shall obtain another equation of the nature of an integral equation representing Fox's H-functions of two variables from (4.1) when both sides are multiplied by $z^{t-1}(1+s)^{k-r}$ where $0 < \operatorname{Re}(t) < \operatorname{Re}(r)$, and integrated with respect to z between 0 and ∞. If t, k, r are so related that a number σ can be chosen which will satisfy both of the inequalities $0 < \sigma < \operatorname{Re}(k)$, $0 < \operatorname{Re}(t - \sigma) < \operatorname{Re}(r-k)$ the order of integration on the right may be inverted to obtain

(5.1)
$$\int_0^\infty \frac{z^{s+t-1}}{(1+z)^{s+r}} H \begin{bmatrix} x\left(\frac{z}{1+z}\right)^h \\ y\left(\frac{z}{1+z}\right)^h \end{bmatrix} dz$$

$$= \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \frac{\Gamma(k-l)}{h^{k-l}} H\left[f\left|{}^{\Delta(h,\, l+s),\ (a_{p_1},\, A_{p_1})}; \ g^{(b_{q_1},\, B_{q_1}),\ \Delta(h,\, k+s)}\right|{}^x_y\right]$$

$$\left\{\int_0^\infty \frac{z^{t-l-1}}{(1+z)^{r-k}}\, dz\right\} dl.$$

Then interpretation of (3.1) and (2.1) to (5.1) finally yields the desired result

$$(5.2)\qquad \frac{\Gamma(r-t)\,\Gamma(r-k)}{h^{r-t}}\,H\left[f\left|\begin{matrix}\Delta(h,t+s),\,(ap_1,Ap_1);\\ g\end{matrix}\right.{}^{(bq_1,Bq_1),\,\Delta(h,r+s)}\left|\begin{matrix}x\\y\end{matrix}\right.\right]$$

$$=\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{\Gamma(k-l)\,\Gamma(t-l)\,\Gamma(r-k-t+l)}{h^{k-l}}$$

$$H\left[f\left|\begin{matrix}\Delta(h,l+s),\,(ap_1,Ap_1);\\ g\end{matrix}\right.{}^{(bq_1,Bq_1),\,\Delta(h,k+s)}\left|\begin{matrix}x\\y\end{matrix}\right.\right]dl$$

provided h being a positive integer >0, $0<\sigma<\mathrm{Re}(k)$, $0<\mathrm{Re}(t-\sigma)<\mathrm{Re}(r-k)$, $p_1\geqslant m_1\geqslant 0$, $q_1\geqslant 0$, $p_2\geqslant m_2\geqslant 0$, $q_2\geqslant n_2\geqslant 0$, $p_3\geqslant m_3\geqslant 0$, $q_3\geqslant n_3\geqslant 0$, $\mathrm{Re}\left(l+s+h\frac{d_i}{D_i}+h\frac{f_j}{F_j}\right)>0\ (1\leqslant i\leqslant n_2;\ 1\leqslant j\leqslant n_3)$, $\lambda_1\leqslant 0$, $\mu_1\leqslant 0$, $\lambda_2>0$, and $\mu_2>0$.

Here we observe some of the restrictions on t, k, r made in the investigation may be relaxed by analytic continuation after we specify that the path of integration is to be curved, if necessary, so that the set of poles due to $\Gamma(r-k-t+l)$ lie to the left and those of $\Gamma(k-l)\Gamma(t-l)$ are to the right. This implies that t, k, r are such that the required path of integration may be found indicating no pole of one set can coincide with a pole of the other set.

6. Applications

This section is devoted to the study of a few very important particular cases of the main result (5.2) on an integral equation incorporating the generalization of numerous formulae on H-and G-functions which are fundamental in the applications since they include as special cases a vast number of the commonly used functions of analysis.

Some interesting cases are exhibited as under:

(i) If we substitute $A_j=B_i=\cdots$ etc. $=1$ $(1\leqslant j\leqslant p_1,\ 1\leqslant i\leqslant q_1,\ \ldots,$ etc.) in (5.2) and then by specific adjustment of the parameters etc., the result involving Meijer's G-functions of two variables can be obtained.

(ii) In (5.2), when we employ $p_1=m_1$, $p_3=m_3$, $n_3=1$, $f_1=0$ and replace p_1+m_2, p_1+p_2, q_1+q_2 and n_2 by m, p, q and n respectively and with proper choice of parameters etc., then allow $y\to 0$, the relation on Fox's H-functions of one variable (generalizations of Meijers' G-functions) can be deduced.

(iii) In (5.2), taking all A's, B's, etc. equal to unity and $m_1=p_1=\lambda$, $q_1=\nu$, $m_2=m_3=p_2=p_3=\mu$, $n_2=n_3=1$, $q_2=q_3=\rho+1$, $d_1=f_1=0$ and replacing a_j, b_j, $1-c_j$, $1-d_j$, $1-e_j$ and $1-f_j$ by α_j, γ_j, β_i, δ_j, β_j', and δ_j' etc. respectively, and then performing (2.5), we obtain an integral equation for Kampé de Fériet functions

$$(6.1)\qquad \frac{\Gamma(r-t)}{h^{r-t}} \prod_{i=0}^{h-1} \left\{ \frac{\Gamma\left(\frac{t+s+i}{h}\right)}{\Gamma\left(\frac{r+s+i}{h}\right)} \right\} F_{\nu+h,\,\rho}^{\lambda+h,\,\mu} \left[\begin{matrix} | \alpha |_\lambda,\ \Delta(h,\, t+s) : | \beta, \beta' |_\mu \\ | \gamma |_\nu,\ \Delta(h,\, r+s) : | \delta, \delta' |_\rho \end{matrix} \middle| \begin{matrix} x \\ y \end{matrix} \right]$$

$$= \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \frac{\Gamma(k-l)\,\Gamma(t-l)\,\Gamma(r-k-t+l)}{h^{k-l}\,\Gamma(r-k)} \prod_{i=0}^{h-1} \left\{ \frac{\Gamma\left(\frac{l+s+i}{h}\right)}{\Gamma\left(\frac{k+s+i}{h}\right)} \right\}$$

$$F_{\nu+h,\,\rho}^{\lambda+h,\,\mu} \left[\begin{matrix} | \alpha |_\lambda,\ \Delta(h,\, l+s) : | \beta, \beta' |_\mu \\ | \gamma |_\nu,\ \Delta(h,\, k+s) : | \delta, \delta' |_\rho \end{matrix} \middle| \begin{matrix} x \\ y \end{matrix} \right] dl$$

where h is a positive integer >0 and $\lambda+\mu<\nu+\rho+1$, $\{\lambda+\mu=\nu+\rho+1;$ then $|x|<1$, $|y|<1\}$ or if $\lambda+\mu+1>\nu+\rho$, then $|\arg y|$, $|\arg x|<(\lambda+\mu-$ $-\nu-\rho)\frac{\pi}{2}$, $\mathrm{Re}\,(l+s)>0$, $0<\sigma<\mathrm{Re}\,(k)$, $0<\mathrm{Re}\,(t-\sigma)<\mathrm{Re}\,(r-k)$.

Special Case. Now, applying (2.6) in (6.1) and then selecting $h=m=1$, $\beta+\beta'=-n$, $\alpha_1=n+1$, $\gamma_1=1$, $t+s=\xi$, $k+s=p$, $r+s=q$, $l+s=\zeta$, we obtain a well-known result on "Integral Equation" by Rice [7, p. 111, (1.10)]:

$$(6.2)\qquad \frac{\Gamma(q-\xi)\,\Gamma(\xi)\,\Gamma(p)\,\Gamma(q-p)}{\Gamma(q)} H_n(\xi,\, q,\, x)$$

$$= \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \Gamma(\zeta)\,\Gamma(p-\zeta)\,\Gamma(\xi-\zeta)\,\Gamma(q-\xi-p+\zeta)\, H_n(\zeta,\, p,\, x)\, d\zeta$$

in which $0<\sigma<\mathrm{Re}\,(p)$, $0<\mathrm{Re}\,(\xi-\sigma)<\mathrm{Re}\,(q-p)$ and $H_n(p,\, q,\, x)={}_3F_2(-n,\, n+1,$ $p;\, 1,\, q;\, x)$ is the Rice's polynomial.

Since H-function of two variables is more general than even Meijer's G-function of two variables which itself is a generalization of many mathematical functions arising in analysis and applied problems, the result investigated here becomes a master or key formula from which a large number of important relations can be deduced for various functions scattered throughout the literature.

REFERENCES

[1] T. J. I.' A. Bromwich, *An introduction to the theory of infinite series*, Macmillan & Co., London, 1931.

[2] A. Erdélyi, *Higher Transcendental functions*, Vol. 1, McGraw-Hill, New York, 1953.

[3] A. Erdélyi, *Tables of integral transforms*, Vol. 1, McGraw-Hill, New York, 1954.

[4] C. Fox, *The G and H functions as symmetrical Fourier Kernels*, Trans. Amer. Math. Soc. 98, (1961), 395—429.

[5] P. C. Munot and S. L. Kalla, *On an extension of generalized function of two variables*, Univ. Nac. Tucuman, Rev. Ser. A. 21 (1971), 67—84.

[6] B. L. Sharma, *On the generalized function of two variables I*, Ann. Soc. Sci. Bruxelles, 79 (1965), 26—40.

[7] S. O. Rice, *Some properties of* 3F2 $(-n, n+1, \varsigma, 1, p; v)$, Duke Math, J., 6 (1940), 108—119.

Manilal Shah, M. Sc., Ph. D., 6/6, Mahatma Gandhi Road, INDORE-1 (M- P.), 452001, INDIA