

EXPANSION OF KAMPÉ DE FÉRIET'S FUNCTION

*Manilal Shah**

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Synopsis

In an attempt to give a unified presentation for generalizations of various interesting results available in papers or elsewhere scattered throughout the literature, the author establishes here a lemma with the help of Mellin and Hankel transforms of Fox's H — function.

It runs as follows:

$$\sum_{\xi=0}^{\infty} \frac{1}{\xi!} H_{q, p+1}^{n+1, m} \left[x \left| \begin{array}{c} \{(1-b_q, B_q)\} \\ \xi + \sigma, \{(1-a_p, A_p)\} \end{array} \right. \right] = x^\sigma \Phi(\sigma),$$

where

$$\Phi(\sigma) = \frac{\prod_{j=1}^m \Gamma(b_j + B_j \sigma) \prod_{j=1}^n \Gamma(1 - a_j - A_j \sigma)}{\prod_{j=m+1}^q \Gamma(1 - b_j - B_j \sigma) \prod_{j=n+1}^p \Gamma(a_j + A_j \sigma)}.$$

It may be of interest to study how this lemma is used in the derivation of the main result on an expansion theorem for Kampé de Fériet's function in the form

$$\begin{aligned} & x^\sigma F_{u, v}^{h, l} \left[\begin{array}{c} |\alpha|_h : |\beta, \beta'|_l; xy \\ |\gamma|_u : |\delta, \delta'|_v; xz \end{array} \right] \\ &= \{\Phi(\sigma)\}^{-1} \sum_{\xi=0}^{\infty} \frac{1}{\xi!} H_{q, p+1}^{n+1, m} \left[x \left| \begin{array}{c} \{(1-b_q, B_q)\} \\ \xi + \sigma \{(1-a_p, A_p)\} \end{array} \right. \right] \\ & F_{u+q, v}^{h+p+1, l} \left[\begin{array}{c} -\xi, |\alpha|_h, (a_p + \sigma) : |\beta, \beta'|_l; (-1)^{m+n+q} y \\ |\gamma|_u, (b_q + \sigma) : |\delta, \delta'|_v, (-1)^{m+n+q} z \end{array} \right] \end{aligned}$$

* Postal Address: Manilal Shah, M. Sc., Ph. D.,
 6/6, M. G. Road, Indore—1 (M. P.), 452—001, India.

Special cases of the theorem include a large number of generalizations due to Meijer, Sharma, Srivastava and many others. Also, it yields several corollaries which might find applications in various problems of Science and Technology.

1. Prerequisites

An extension of Meijer's G -function [4, p. 207 (1)] is given by Fox [8, p. 408], viz,

$$(1.1) \quad H_{p,q}^{m,n} \left[x \left| \begin{matrix} \{(a_p, A_p)\} \\ \{(b_q, B_q)\} \end{matrix} \right. \right] = \frac{1}{2\pi i} \int_C \psi(\xi) x^\xi d\xi,$$

where

$$\psi(\xi) = \frac{\prod_{j=1}^m \Gamma(b_j - B_j \xi) \prod_{j=1}^n \Gamma(1 - a_j + A_j \xi)}{\prod_{j=m+1}^q \Gamma(1 - b_j + B_j \xi) \prod_{j=n+1}^p \Gamma(a_j - A_j \xi)}.$$

For more details to H -function, we refer the reader to the excellent work of Fox and Braaksma [2 and 8].

A generalized hypergeometric function of two variables due to Kampé de Fériet J. [1, p. 150] has been defined by Srivastava and Saran [11, p. 435] in computable notation

$$(1.2) \quad F_{n,p}^{m,l} \left[\begin{matrix} a|_m : |b, b'|_l; x \\ c|_n : |d, d'|_p; y \end{matrix} \right] = \sum_{r,s=0}^{\infty} \frac{\Pi(a_m)_{r+s} \Pi\{(b_l)_r (b'_l)_s\} x^r y^s}{r! s! \Pi(c_n)_{r+s} \Pi\{(d_p)_r (d'_p)_s\}}.$$

In particular, if parameters in (1.2) are specialized and with the aid of [1, p. 14, (125)], the Appell function F_2 and its confluent function are given by

$$(1.3) \quad F_{0,1}^{1,1} \left[\begin{matrix} a : b, b'; x \\ - : d, d'; y \end{matrix} \right] = F_2(a : b, b'; d, d'; x, y),$$

$$(1.4) \quad \psi_2(a, d, d'; x, y) = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n} x^m y^n}{(d)_m (d')_n m! n!}.$$

Again, from [1, p. 132, (133)]:

$$(1.5) \quad M_{k,\mu,\nu}(x, y) = x^{\mu+\frac{1}{2}} y^{\nu+\frac{1}{2}} e^{-\frac{1}{2}(x+y)} \psi_2(\mu+\nu-k+1, 2\mu+1, 2\nu+1; x, y),$$

and

$$(1.6) \quad \lim_{\varepsilon \rightarrow 0} M_{k,\mu,\varepsilon-\frac{1}{2}}(x, \varepsilon^2 y) = M_{k,\mu}(x)$$

where $M_{k,\mu,\nu}(x, y)$ and $M_{k,\mu}(x)$ are the Whittaker functions in two and one variables respectively.

Recently the author, in a sequence of papers [15—20], has investigated Generalization, Co-ordination and Co-relation of many results in Special Functions, Integral Transforms, Operational Calculus and Integral Equations etc.; in an extension of this work, the present paper incorporates a systematic achievement for the elegant unification of certain formulas.

Section 2 contains preliminary results and definitions, Section 3 establishes a Lemma by making use of Mellin and Hankel transforms of Fox's H -functions and Section 4 proves the derivation of our main result. Several examples are presented in Section 5.

Often, as a space saver and printing convenience we have frequently employed the shorthand notations:

(i) $\{(a_p, A_p)\}$ represents the set of p -ordered pairs:

$$(a_1, A_1), \dots, (a_j, A_j); (a_{j+1}, A_{j+1}), \dots, (a_p, A_p);$$

(ii) $|a|_m$ or a_m stands for the sequence of elements:

$$a_1, a_2, \dots, a_j, \dots, a_m;$$

(iii) $\Pi(a_m)_n$ denotes $(a_1)_n (a_2)_n \dots (a_m)_n$;

(iv) $\Delta(m, n) = \frac{n}{m}, \frac{n+1}{m}, \dots, \frac{n+m-1}{m}$;

(v) $(a_p + m) = a_1 + m, a_2 + m, \dots, a_p + m$.

2 Preliminary definitions and results

(i) The Mellin transform [6, p. 305]:

$$(2.1) \quad g(s) = M\{f(x); s\} = \int_0^{\infty} f(x) x^{s-1} dx.$$

(ii) The Hankel transform [7, p. 3]:

$$(2.2) \quad g(y; \nu) = H_{\nu}\{f(x); y\} = \int_0^{\infty} f(x) J_{\nu}(xy) (xy)^{\frac{1}{2}} dx.$$

(iii) The Mellin transform of Fox's H -function [10, p. 159]:

$$(2.3) \quad M\left\{H_{u,v}^{d,e}\left[\mu x^{\lambda} \left| \begin{matrix} (\alpha_u, \eta_u) \\ (\beta_v, \rho_v) \end{matrix} \right. \right]; s\right\} \\ = \frac{1}{\lambda} \frac{\prod_{j=1}^d \Gamma\left(\beta_j + \frac{\rho_j}{\lambda} s\right) \prod_{j=1}^e \Gamma\left(1 - \alpha_j - \frac{\eta_j}{\lambda} s\right)}{\prod_{j=d+1}^v \Gamma\left(1 - \beta_j - \frac{\rho_j}{\lambda} s\right) \prod_{j=e+1}^u \Gamma\left(\alpha_j - \frac{\eta_j}{\lambda} s\right)} \mu^{-\frac{s}{\lambda}}$$

provided

$$(a) \quad -\min_{1 \leq j \leq d} R\left(\frac{\beta_j}{\rho_j}\right) < R(s) < \frac{\lambda}{\eta_j} - \lambda \max_{1 \leq j \leq e} R\left(\frac{\alpha_j}{\eta_j}\right);$$

and

$$(b) \quad \begin{cases} \theta > 0, & |\arg \mu| < \frac{1}{2} \theta \pi, \\ \theta \geq 0, & |\arg \mu| \leq \frac{1}{2} \theta \pi \text{ and } R(\Phi + 1) < 0, \end{cases}$$

where

$$\theta = \sum_1^e \eta_j - \sum_{e+1}^u \eta_j + \sum_1^d \rho_j - \sum_{d+1}^v \rho_j;$$

$$\Phi = \frac{1}{2} (u - v) + \sum_1^v \beta_j - \sum_1^u \alpha_j.$$

(iv) The Hankel transform of Fox's H -function due to Shah [14, p. 7, (3.2)]:

$$(2.4) \quad H_\nu \left\{ x^\mu H_{p,q}^{n,m} \left[\alpha x^{2\rho} \left| \begin{matrix} \{(a_p, A_p)\} \\ \{(b_q, B_q)\} \end{matrix} \right. \right]; y \right\}$$

$$= \frac{(\alpha \rho)^{\mu+1/2}}{y^{\mu+1}} H_{p+2\rho,q}^{n,m+\rho} \left[\alpha \left(\frac{2\rho}{y} \right)^{2\rho} \left| \psi \right. \right]$$

where

$$\psi = \Delta \left(\rho, -\frac{1}{2} \mu - \frac{1}{2} \nu + \frac{1}{4} \right), \{(a_p, A_p)\}, \Delta \left(\rho, \frac{1}{2} \nu - \frac{1}{2} \mu + \frac{1}{4} \right);$$

$$\{(b_q, B_q)\}$$

ρ is a positive integer > 0 ; y is taken to be a positive real variable; and

$$\begin{cases} (i) & 0 \leq m \leq p, 0 \leq n \leq q; \\ (ii) & \mathcal{F}_1 = \sum_1^b A_j - \sum_1^q B_j \leq 0; \\ (iii) & \mathcal{F}_2 = \sum_1^m A_j - \sum_{m+1}^p A_j + \sum_1^n B_j - \sum_{n+1}^q B_j > 0, |\arg \alpha| < \frac{1}{2} \pi \mathcal{F}_2; z \\ (iv) & -\operatorname{Re}(\nu) - \frac{3}{2} < \operatorname{Re} \left[\mu + 2\rho \frac{b_j}{B_j} \right] < -1. \end{cases}$$

(v) Series of Bessel function [5, p. 99, (9)]:

$$(2.5) \quad x^\nu = 2^\nu \Gamma \left(1 + \frac{1}{2} \nu \right) \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{1}{2} x \right)^{\frac{1}{2} \nu + n} J_{\frac{1}{2} \nu + n}(x).$$

(vi) A simple identity on H -functions:

$$(2.6) \quad H_{p,q}^{m,n} \left[x^{-1} \left| \begin{matrix} \{(a_p, A_p)\} \\ \{(b_q, B_q)\} \end{matrix} \right. \right] = H_{q,p}^{n,m} \left[x \left| \begin{matrix} \{(1-b_q, B_q)\} \\ \{(1-a_p, A_p)\} \end{matrix} \right. \right];$$

3. The Requisite Lemma

We consider here a lemma involving Fox's H -function.

L e m m a: *If*

(i) $0 \leq m \leq q, 0 \leq n \leq p$:

(ii) $\lambda \equiv \sum_1^q B_j - \sum_1^p A_j + 1 \leq 0$;

(iii) $\mu = \sum_1^m B_j - \sum_{m+1}^q B_j + \sum_1^n A_j - \sum_{n+1}^p A_j + 1 > 0$; $|\arg x| < \frac{1}{2} \pi \mu$.

(iv) $\operatorname{Re} \left(\sigma + \frac{1-a_j}{A_j} \right) > 0$; and $\operatorname{Re} \left(\sigma - \frac{b_i}{B_i} \right) < 0$;

Then

$$(3.1) \quad \sum_{\xi=0}^{\infty} \frac{1}{\xi!} H_{q,p+1}^{n+1,m} \left[x \left| \begin{matrix} \{(1-b_q, B_q)\} \\ \xi + \sigma, \{(1-a_p, B_p)\} \end{matrix} \right. \right] = x^{\sigma} \Phi(\sigma)$$

where

$$\Phi(\sigma) = \frac{\sum_{j=1}^m \Gamma(b_j + B_j \sigma) \sum_{j=1}^n \Gamma(1 - a_j - A_j \sigma)}{\sum_{j=m+1}^q \Gamma(1 - b_j - B_j \sigma) \sum_{j=n+1}^p \Gamma(a_j + A_j \sigma)}.$$

Proof: In consequence of (2.5), we first replace x, ν by \mathfrak{z} and 2σ respectively. Thus, it is equivalent to

$$(3.2) \quad \mathfrak{z}^{\sigma} = \Gamma(\sigma + 1) \sum_{\xi=0}^{\infty} \frac{1}{\xi!} \mathfrak{z}^{\xi} J_{\sigma+\xi}(2\mathfrak{z}).$$

Next multiply both sides of (3.2) by

$$\mathfrak{z}^{\sigma-1} H_{p,q}^{m,n} \left[\frac{\mathfrak{z}^2}{x} \left| \begin{matrix} \{(a_p, A_p)\} \\ \{(b_q, B_q)\} \end{matrix} \right. \right],$$

and by integration, it gives

$$(3.3) \quad \int_0^{\infty} \mathfrak{z}^{2\sigma-1} H_{p,q}^{m,n} \left[\frac{\mathfrak{z}^2}{x} \left| \begin{matrix} \{(a_p, A_p)\} \\ \{(b_q, B_q)\} \end{matrix} \right. \right] d\mathfrak{z} \\ = \Gamma(\sigma + 1) \sum_{\xi=0}^{\infty} \frac{1}{\xi!} \int_0^{\infty} \mathfrak{z}^{\sigma+\xi-1} J_{\sigma+\xi}(2\mathfrak{z}) H_{p,q}^{m,n} \left[\frac{\mathfrak{z}^2}{x} \left| \begin{matrix} \{(a_p, A_p)\} \\ \{(b_q, B_q)\} \end{matrix} \right. \right] d\mathfrak{z}.$$

The change in order of integration and summation in the above process seems to be most thoroughly justified in view of de la Vallée Poussin's theorem [3, p. 500] and by the principle of analytic continuation subject to the conditions imposed with (3.1). Moreover, Fox's H -function is also an analytic function.

Now employ (2.3) and (2.4) with obvious changes of parameters in (3.3).

Finally, the performance of (2.6) in the latter yields (3.1) after a bit of simplification.

4. The Main result

In this section we derive an expansion theorem for Kampé de Fériet's function in a series involving product of a Fox's H -function and generalized hypergeometric function of two variables.

It indicates as under:

Theorem: *Let*

- (i) $h + l \leq u + v + 1$;
- (ii) $h + p + l + 1 \leq u + q + v$;
- (iii) $|xy| < 1$, $|x\delta| < 1$.

Then

$$(4.1) \quad x^\sigma F_{u,v}^{h,l} \left[\begin{matrix} |\alpha|_h : |\beta, \beta'|_l; xy \\ |\gamma|_u : |\delta, \delta'|_v; x\delta \end{matrix} \right] \\ = \{\Phi(\sigma)\}^{-1} \sum_{\xi=0}^{\infty} \frac{1}{\xi!} H_{q,p+1}^{n+1,m} \left[x \left| \begin{matrix} \{(1-b_q, B_q)\} \\ \xi + \sigma, \{(1-a_p, A_p)\} \end{matrix} \right. \right] \\ F_{u+q,v}^{h+p+1,l} \left[\begin{matrix} -\xi, |\alpha|_h, (a_p + \sigma) : |\beta, \beta'|_l; (-1)^{m+n-q} y \\ |\gamma|_u, (b_q + \sigma) : |\delta, \delta'|_v; (-1)^{m+n-q} \delta \end{matrix} \right]$$

provided

- (iv) $0 \leq m \leq q$, $0 \leq n \leq p$;
- (v) $\lambda \equiv \sum_1^q B_j - \sum_1^p A_j + 1 \leq 0$;
- (vi) $\mu = \sum_1^m B_j - \sum_{m+1}^q B_j + \sum_1^n A_j - \sum_{n+1}^p A_j + 1 > 0$; $|\arg x| < \frac{1}{2} \pi \mu$;
- (vii) $\operatorname{Re} \left(\sigma + \frac{1-a_j}{A_j} \right) > 0$; and $\operatorname{Re} \left(\sigma - \frac{b_i}{B_i} \right) < 0$.

Proof. The above theorem is easily proved when the result (3.1) of 3 is referred to the power series expansions of Kampé de Fériet's series of two variables on the lefthand side of (4.1) and then equate the terms associated with the same Fox's H -functions. The analysis given here is purely formal. Therefore, we omit such details.

5. Applications

This section is devoted for illustration on important corollaries of our main result (4.1) which will give rise to extensions of some well-known formulas.

However, a few of them are mentioned below:

(i) Evidently, (4.1) provides a generalization of Sharma's formula [13, pp. 34—35, (6)] to which it would reduce for $A_j = B_i = 1$ ($1 \leq j \leq p$, $1 \leq i \leq q$).

(ii) On the other hand, when we select $\alpha = \gamma$ and $h = u$, we find an interesting result on product of two generalized hypergeometric functions

$$\begin{aligned}
 (5.1) \quad & x^\sigma {}_lF_v \left(\begin{matrix} \beta_l \\ \delta_v \end{matrix}; xy \right) {}_lF_v \left(\begin{matrix} \beta'_l \\ \delta'_v \end{matrix}; xz \right) \\
 & = \{ \Phi(\sigma) \}^{-1} \sum_{\xi=0}^{\infty} \frac{1}{\xi!} H_{q,p+1}^{n+1,m} \left[x \middle| \begin{matrix} \{(1-b_q, B_q)\} \\ \sigma + \xi, \{(1-a_p, A_p)\} \end{matrix} \right] \\
 & \quad F_{q,v}^{p+1,l} \left[\begin{matrix} -\xi, (a_p + \sigma); |\beta, \beta'_l; (-1)^{m+n-q} y \\ (b_q + \sigma); |\delta, \delta'_v; (-1)^{m+n-q} z \end{matrix} \right].
 \end{aligned}$$

Again, for $A_j = B_i = 1$ ($1 \leq j \leq p$; $1 \leq i \leq q$), $n = 0$, $p = 1$, $m = q = 0$ and application of [4, p. 216, (3)], our formula (5.1) leads us to a familiar result due to Srivastava [12, p. 246, (22)].

(iii) If in (4.1), we take $\beta = \delta$, $\beta' = \delta'$, $l = v$ and then with simplification, we shall get a useful formula

$$\begin{aligned}
 (5.2) \quad & x^\sigma {}_hF_u \left(\begin{matrix} \alpha_h \\ \gamma_u \end{matrix}; xy \right) \\
 & = \{ \Phi(\sigma) \}^{-1} \sum_{\xi=0}^{\infty} \frac{1}{\xi!} H_{q,p+1}^{n+1,m} \left[x \middle| \begin{matrix} \{(1-b_q, B_q)\} \\ \sigma + \xi, \{(1-a_p, A_p)\} \end{matrix} \right] \\
 & \quad {}_{h+p+1}F_{u+q} \left[\begin{matrix} -\xi, \alpha_h, (a_p + \sigma); \\ \gamma_u, (b_q + \sigma) \end{matrix}; (-1)^{m+n-q} y \right].
 \end{aligned}$$

Further, taking all A 's and B 's equal to unity and using [4, p. 215, (1)], (5.2) can be transformed into a well-known result which was proved earlier by Meijer [9, p. 311, (237)].

(iv) An important instructive case occurs if each A_j and B_j is unity and $m = q = 0$, $n = 1$, $p = 1$, $a_1 = 1$ e.c., and by virtue of [4, p. 216, (4)] in (4.9) leads to

$$\begin{aligned}
 (5.3) \quad & x^\sigma F_{u,v}^{h,l} \left[\begin{matrix} |\alpha|_h; |\beta, \beta'_l; x^2 y \\ |\gamma|_u; |\delta, \delta'_v; x^2 z \end{matrix} \right] \\
 & = \frac{1}{2} \{ \Gamma(-\sigma) \}^{-1} \sum_{\xi=0}^{\infty} \frac{1}{\xi!} x^\xi k_{\xi+\sigma}(2x) \\
 & \quad F_{u,v}^{h+2,l} \left[\begin{matrix} -\xi, |\alpha|_h, 1 + \sigma; |\beta, \beta'_l; -y \\ |\gamma|_u, \quad \quad \quad : |\delta, \delta'_v; -z \end{matrix} \right]
 \end{aligned}$$

where $k_\nu(x)$ is the modified Bessel function of the third kind.

(v) Yet the other illustrative deductions of (4.9) would follow immediately by making suitable substitutions on the basis of (1.5) and (1.6). We omit their descriptions.

Conclusion: Since Fox's H -and Kampé de Fériet's functions are more general than even many Higher Transcendental Functions arising in numerous applications, the result proved above becomes a master or key-formula from which a vast body of special functions can be derived.

REFERENCES

- [1] Appel Paul, and Kampé de Fériet, J., *Fonctions Hypergéométric et Hyper-sphériques; Polynomes d'Hermîtes*, Gauthier — Villars, Paris, 1926.
- [2] Braaksma, B. L. J., *Asymptotic expansion and analytic continuations for a class of Barnes — integrals*. *Compositio Math.* 15 (1963), 239—341.
- [3] Bromwich, T. J. I'a., *An Introduction to the Theory of Infinite Series*, Macmillan 8 Co., Ltd., New York, 1959.
- [4] Erdélyi, A., *Higher Transcendental Functions*, Vol. I, McGraw—Hill, New York, 1953.
- [5] Erdélyi, A., *Higher Transcendental Functions*, Vol. II, McGraw—Hill, New York, 1953.
- [6] Erdélyi, A., *Tables of Integral Transforms*, Vol. I, McGraw—Hill, New York, 1954.
- [7] Erdélyi, A., *Tables of Integral Transforms*, Vol. II, McGraw—Hill, New York, 1954.
- [8] Fox C., *G and H-functions as symmetrical Fourier—Kernels*, *Trans. Math. Soc.* 98 (1962), 395—429; MR 24#A 1427.
- [9] Meijer, C. S., *Expansion theorems for G-function*. *Nederl. Akad. Wetensche Proc. Ser. A.*, 58, № 3 and *Indag Math.*, Vol. 17, № 3 (1955), 309—314.
- [10] Singh Rattan, *Two Theorems on H-Function of Fox*, *Proc. Nat. Acad. Sci, India*, 38 (1968), 155—160; MR 40#5923.
- [11] Srivastava, G., P. and Saran S., *A theorem on Kampé de Fériet function*, *Proc. Camb. Philos. Soc.* 64 (1968), 435—437 MR 36#5404.
- [12] Srivastava, H. M., *Some expansions in products of hypergeometric functions*. *Proc. Camb. Philos. Soc.*, 62 (1966).
- [13] Sharma, B. L., *An expansion Formula for Hypergeometric Functions of Two Variables*, *I. Math. Tokushima Univ.* Vol. 4 (1970), 33—36.
- [14] Shah Manilal, *Various Generalizations on Generalized Functions*, *Portugal Mathematica* (In Press).
- [15] Shah Manilal, *On some applications related to Fox's H-functions of two variables*, *Publ. Inst. Math. (Beograd)*, (N. S.) 16 (30), (1973), 01—08.
- [16] Shah Manilal, *On generalized Meijer function of two variablas and some applications*, *Comment. Math. Univ. St. Paul* 19 (1971), 93—122; MR 45#2223.
- [17] Shah Manilal, *Two integral transform pairs involving H-functions*, *Glasnik Matematički* 7 (27), (1972), 57—65.
- [18] Shah Manilal, *On some relations on H-functions associated with orthogonal polynomials*, *Mathematica Scandinavica*, 30 (1972), 331—336.
- [19] Shah Manilal, *On generalizations of some results and their applications*. *Collectanea Mathematica*, 24 (1973), 249—266.
- [20] Shah Manilal, *Some results on generalized hypergeometric polynomials*, *Proc. Camb. Philos. Soc.* 66 (1969), 95—104; MR 40#403.