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BERNSTEIN'S CONSTANT AND BEST APPROXIMATION ON $[0, \infty)$

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There is a famous result of Bernstein in the Theory of Approximation for which a simple proof has yet to be found. If P_n is the polynomial of best approximation to $|x|$ on $[-1, 1]$, of degree $\leq n$, and

$$E_n = \max_{-1 \leq x \leq 1} \left| |x| - P_n(x) \right|,$$

then according to that result of Bernstein, $\lim_{n \to \infty} nE_n = \mu$ exists and $\mu = 0.282 \pm \pm 0.004$. [1]. Bernstein noticed also, as an interesting coincidence that $1/2\sqrt{\pi} = 0.282094\ldots$, and that it would be interesting to find out whether the constant μ is a new transcendental number, or $\mu = 1/2\sqrt{\pi}$. Bernstein's computation of the approximate value of μ is very complex. It seems that now with the use of computers more precise computations of μ might indicate whether the conjecture $\mu = 1/2\sqrt{\pi}$ is true or false. But before the actual computations could be carried out, it was necessary to obtain a number of purely approximation theoretical results.

A careful analysis of Bernstein's work shows that $\mu/2 \leq B_n(\varphi)$, where

$$B_n(\varphi) = \inf_{a_0, \ldots, a_n} \sup_{t \geq 0} \left| \left(\varphi(t) - \left(a_o + \sum_{k=1}^{n} \frac{a_k}{4t^2 - (2k-1)^2} \right) \right) \cos \pi t \right|$$

and

$$\varphi(x) = x \int_0^1 \frac{t^{x-1/2}}{t+1} \, dt,$$

and that $\lim_{n \to \infty} B_n(\varphi) = \mu/2$ (see [1], p. 52).

In view of this result the approximate computation of μ is reduced to the computation of $B_n(\varphi)$. This computation could be essentially simplified if we could find an extremal function

$$R_n^*(t) = b_0 + \sum_{k=1}^{n} \frac{b_k}{4t^2 - (2k-1)^2}$$

which has the property that

$$B_n(\varphi) = \sup_{t\geq 0} |(\varphi(t) - R_n^*(t)) \cos \pi t|.$$

Once we have enough information about this extremal function R_n^*, in particular, once we know the alternation and uniqueness properties of R_n^*, analog to the well-known properties of polynomials of best uniform approximation on finite intervals, we may be able to construct algorithms for the numerical computation of the constants $B_n(\varphi)$.

The function φ can be expressed also as an infinite integral

$$\varphi(x) = \frac{1}{2} \int_0^\infty e^{-t} (\cos h (t/2x))^{-1} dt$$

so that φ is clearly an increasing function on $[0, \infty)$ with $\varphi(0) = 0$ and $\lim_{x\to\infty} \varphi(x) = \varphi(\infty) = 1/2$.

In this paper we shall consider, more generally, a non-decreasing and continuous function f on $[0, \infty)$ with $f(0) = 0$ and $0 < f(\infty) < \infty$. The family of rational functions $\mathcal{R}_n$ is defined by

$$R \in \mathcal{R}_n \Leftrightarrow R(t) = y_0 + \sum_{k=1}^n \frac{y_k}{4t^2 - (2k-1)^2}.$$

Finally, the constant $B_n(f)$ is defined by

$$B_n(f) = \inf_{R\in\mathcal{R}_n} \sup_{t\geq 0} |(f(t) - R(t)) \cos \pi t|.$$

By using standard arguments, it is easy to see that there always exists an extremal function

$$R_n^*(t) = x_{on} + \sum_{k=1}^n \frac{x_{kn}}{4t^2 - (2k-1)^2}$$

in $\mathcal{R}_n$ such that

$$B_n(f) = \sup_{t\geq 0} |(f(t) - R_n^*(t)) \cos \pi t|.$$

Any point t_0 such that

$$B_n(f) = |(f(t_o) - R_n^*(t_0)) \cos \pi t_0|$$

will be called, as usual, an extremal point.

Next, we shall study the alternation properties of the extremal functions. These properties can be best illustrated by considering the graph of the error function

$$g_n(t) = (f(t) - R_n^*(t)) \cos \pi t.$$

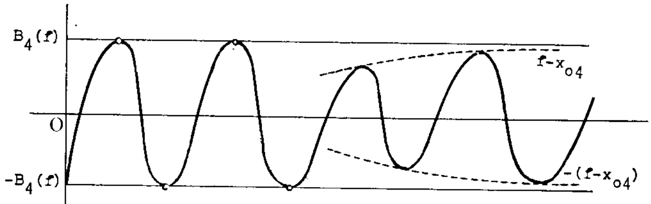

Fig. 1.
Error curve g_4

Beginning with the point $t=0$, the graph of the error function g_n has $n+1$ alternating extremal points in $[0, n]$. It has also the point ∞ as an extremal point in the sense that

$$\liminf_{t\to\infty} g_n(t) = -B_n(t) \quad \text{and} \quad \limsup_{t\to\infty} g_n(t) = B_n(f).$$

The proof of this result seems to be very complex, except when $n=0$. In that case we have

$$B_0(f) \geqslant |f(0)-x_{00}| \geqslant x_{00} \quad \text{and} \quad B_0(f) \geqslant |f(\infty)-x_{00}| \geqslant f(\infty)-x_{00}$$

so that $B_0(f) \geqslant f(\infty)/2$. But, by monotonicity of f,

$$B_0(f) \leqslant \sup_{t\geqslant 0} \left| \left(f(t) - \frac{f(\infty)}{2} \right) \cos \pi t \right| \leqslant \sup_{t\geqslant 0} \left| f(t) - \frac{f(\infty)}{2} \right| = f(\infty)/2.$$

Thus

(1) $$B_0(f) = \frac{1}{2} f(\infty).$$

It is also clear that

(2) $$\overset{*}{R}_0(x) = x_{00} = \frac{1}{2} f(\infty).$$

The most difficult part of the proof of the alternation property in the general case consists in showing that the extremal function $\overset{*}{R}_n$ has non-negative coefficients. An elementary proof of that result would simplify considerably the proof of the alternation theorem for any positive integer n. We mention the following inequalities as a first step in that direction:

(3) $$x_{on} \geqslant \frac{1}{2} f(\infty)$$

and

(4) $$\sum_{k=1}^{n} \frac{x_{kn}}{(2k-1)^2} \geqslant 0.$$

We have first

$$\frac{1}{2} f(\infty) = B_0(f) \geqslant B_n(f) \geqslant |f(\infty) - x_{on}| \geqslant f(\infty) - x_{on}$$

and (3) follows.

Next

$$-x_{on}+\sum_{k=1}^{n}\frac{x_{kn}}{(2k-1)^2}=$$

$$=f(0)-R_n^*(0)\geqslant-|f(0)-R_n^*(0)|\geqslant-B_n(f)\geqslant-\frac{1}{2}f(\infty)$$

and (4) follows from (3).

The fact that the coefficients of the extremal function should be positive was observed already by Bernstein in his computation of $B_3(\varphi)$.

Another result which is very important from the computational point of view is the analog of a well-known theorem of de la Vallée Poussin [2]. This result says that if we can find an error function

$$g_n(t)=\left(f(t)-y_0-\sum_{k=1}^{n}\frac{y_k}{4t^2-(2k-1)^2}\right)\cos\pi t,$$

with $y_0<f(\infty)$, such that

$$g_n(t_k)=(-1)^{k+1}\lambda_k,\quad(\lambda_k>0),\ k=0,\ 1,\ 2,\ldots,\ n$$

where $t_0=0$, $t_k\in[k-1,\ k]$, $k=1,\ldots,\ n$, and

$$\limsup_{t\to\infty}|g_n(t)|=\lambda_\infty>0,$$

then

$$B_n(f)\geqslant\min(\lambda_0,\ \lambda_1,\ldots,\ \lambda_n,\ \lambda_\infty).$$

In Section 1 we shall prove the alternation and uniqueness properties of the extremal function R_n^* for $n=1$. In Section 2 we shall prove the analog of the de la Vallée Poussin's theorem also for $n=1$. In a later paper we will show that all these results are valid for any positive integer n.

As we have pointed out earlier, the basic motivation for this work is the problem of numerical evaluation of the Bernstein's constant μ. Since the constant $\mu/2$ is the limit of the sequence $(B_n(\varphi))$, another problem which will be considered in a later paper is to compute the terms of this sequence for sufficiently large values of n. Although it is not reasonable to expect that the first two terms of the sequence $(B_n(\varphi))$ would give a close approximation to $\mu/2$, it is interesting to observe that $B_0(\varphi)=0.25$ and that from the results of this paper it follows that

(5) $$B_1(\varphi)=0.15149080\ldots$$

A good approximation to the extremal error function is the function

$$g(t)=\left(\varphi(t)-0.3450919728-\frac{0.1901839456}{4t^2-1}\right)\cos\pi t.$$

As it is easy to verify, the error function g has the following properties:

$$\sup_{[0,\ \infty)}|g(t)|\leqslant0.1549080307$$

and also

$$g(0)=-0.1549080272\ldots$$

$$g(0.43)=0.1549080307\ldots$$

and

$$\limsup_{t\to\infty} |g(t)| = 0.1549080272\ldots$$

Hence we can conclude that $B_1(\varphi) \leqslant 0.1549080307\ldots$ On the other hand, in view of the analog of the de la Vallée Poussin's result, we have $B_1(\varphi) \leqslant \leqslant 0.1549080272$ and (5) follows.

1. Positivity of coefficients, alternation property and uniqueness for $n=1$. Using notation from the preceding section, we shall consider here a non-decreasing and continuous function f on $[0, \infty)$ with $f(0)=0$ and $0<f(\infty)<\infty$, the constant

$$B_1(f) = \inf_{R\in\mathcal{R}_1} \sup_{t\geqslant 0} |(f(t)-R(t))\cos\pi t| \tag{1.1}$$

and the extremal function $R^* \in \mathcal{R}_1$ defined by

$$B_1(f) = \sup_{t\geqslant 0} |(f(t)-R^*(t))\cos\pi t|, \tag{1.2}$$

By (3) and (4) we see immediately that the extremal function

$$R^*(t) = x_0 + \frac{x_1}{4t^2-1}$$

has non-negative coefficients. More precisely, we have $x_0 \geqslant f(\infty)/2 > 0$ and $x_1 \geqslant 0$.

We shall always write $g(t)=h(t)\cos\pi t$, where $h(t)=f(t)-R^*(t)$. Then from (1.2) follows that $|g(t)| \leqslant B_1(f)$ for all $t\in[0, \infty)$. We shall often use the inequality

$$|h(\infty)| = |f(\infty)-x_0| \leqslant B_1(f) \tag{1.3}$$

which follows from the fact that $|h(k)|=|g(k)| \leqslant B_1(f)$ for every $k=1, 2, \ldots$

The main result of this section may be stated as follows.

Theorem 1. *The extremal error function g has the following alternation property*

(i) $$g(0) = -B_1(f),$$

(ii) $$g(c) = B_1(f) \quad \textit{for a} \quad c\in(0, 1],$$

(iii) $$\liminf_{t\to\infty} g(t) = -B_1(f) \quad \textit{and} \quad \limsup_{t\to\infty} g(t) = B_1(f).$$

The extremal function R^* is uniquely determined. We have

$$x_0 = f(\infty) - B_1(f).$$

$$x_1 = f(\infty) - 2B_1(f).$$

For the proof of Theorem 1 we need the following lemma.

Lemma 1.1. *If $x_1=0$, then $f(1)=0$.*

Proof of Lemma 1.1. Since f is non-decreasing and $f(0)=0$, we always have $f(1)\geqslant 0$. Suppose that $x_1=0$ and $f(1)>0$. Then by (1) and (2)

$$B_1(f) = B_0(f) = \frac{1}{2} f(\infty) = B$$

and

$$R^*(t)=\frac{1}{2}f(\infty)=B$$

so that $g(t)=h(t)\cos\pi t$, where $h(t)=f(t)-B$. We have clearly $|h(t)|\leqslant B$ for all $t\in[0,\infty)$. Since $g(0)=-B<0$, $g(1)=B-f(1)<B$ and $|g(t)|\leqslant B|\cos\pi t|<B$ for $0<t<1$, we have

$$\max_{[0,\,1]} g(t)=M<B. \tag{1.4}$$

Let

$$\tilde{h}(t)=h(t)-\alpha\left(1+\frac{3}{4t^2-1}\right)=f(t)-\left(B+\alpha+\frac{3\alpha}{4t^2-1}\right)$$

and

$$\tilde{g}(t)=\tilde{h}(t)\cos\pi t$$

where α is chosen so that

$$0<\alpha<\frac{B-M}{3\pi}. \tag{1.5}$$

Since for $0\leqslant t\leqslant 1$

$$0<-\left(1+\frac{3}{4t^2-1}\right)\cos\pi t\leqslant(4t^2+2)\left|\frac{\cos\pi t}{4t^2-1}\right|\leqslant 3\pi$$

we have, in view of (1.4) and (1.5), on [0, 1],

$$-B\leqslant g(t)<\tilde{g}(t)=g(t)-\alpha\left(1+\frac{3}{4t^2-1}\right)\cos\pi t$$

$$<M+\left(\frac{B-M}{3\pi}\right)3\pi=B.$$

Consequently, by the continuity of g it follows that

$$\max_{[0,\,1]}|\tilde{g}(t)|<B. \tag{1.6}$$

The fact that the function $\tilde{g}$ satisfies the same inequality on $[1,\infty)$ follows from the monotonicity of $\tilde{h}$. We have, on $[1,\infty)$,

$$\tilde{h}(t)\leqslant\tilde{h}(\infty)=f(\infty)-B-\alpha=B-\alpha. \tag{1.7}$$

On the other hand, by (1.6) we have

$$\tilde{h}(t)\geqslant\tilde{h}(1)=-g(1)>-B. \tag{1.8}$$

From (1.7) and (1.8) follows that

$$\sup_{[1,\,\infty)}|\tilde{g}(t)|<B. \tag{1.9}$$

Finally, from (1.6) and (1.9) we conclude that

$$\sup_{[0,\,\infty)}|\tilde{g}(t)|<B=B_1(f),$$

which is impossible, Hence, Lemma 1.1 is proved.

Proof of Theorem 1. Suppose first that $x_1=0$. Then $B_1(f)=B_0(f)=f(\infty)/2$, $R^*(t)=x_0=f(\infty)/2$ and $g(t)=(f(t)-f(\infty)/2)\cos\pi t$. Hence

$$g(0)=-f(\infty)/2=-B_1(f).$$

By Lemma 1.1 we have $f(1)=0$ and so

$$g(1)=(f(1)-f(\infty)/2)\cos\pi=f(\infty)/2=B_1(f).$$

Finally, we have

$$\liminf_{t\to\infty} g(t)=-f(\infty)/2=-B_1(f)$$

and

$$\limsup_{t\to\infty} g(t)=f(\infty)/2=B_1(f).$$

Hence, Theorem 1 is proved if $x_1=0$.

Next, suppose that $x_1>0$.

To prove the alternation property (i) suppose the contrary: $-B_1(f)<g(0)=h(0)$. Put

$$\tilde h(t)=h(t)-\alpha+\frac{4\alpha}{4t^2-1}=f(t)-x_0+\alpha-\frac{x_1-4\alpha}{4t^2-1}$$

and $\tilde g(t)=\tilde h(t)\cos\pi t$. The number α is chosen so that

$$0<\alpha<\min(x_1/4,\ (h(0)+B_1(f))/5,\ B_1(f)).$$

By this choice of α we have $\tilde h$ strictly increasing on [0, 1/2) and (1/2, ∞). Therefore

$$-B_1(f)<h(0)-5\alpha=\tilde h(0)\leqslant\tilde h(t) \tag{1.10}$$

on [0, 1/2), and, by (1.3),

$$\tilde h(t)\leqslant\tilde h(\infty)=h(\infty)-\alpha\leqslant B_1(f)-\alpha \text{ on } (1/2,\ \infty). \tag{1.11}$$

Now, on [0, 1/2) we have, by (1.10),

$$\tilde g(t)=\tilde h(t)\cos\pi t>-B_1(f)\cos\pi t\geqslant -B_1(f)$$

and on (1/2, 1] we have, by (1.11),

$$\tilde g(t)=\tilde h(t)\cos\pi t\geqslant-(B_1(f)-\alpha).$$

Since $g(1/2)=(x_1-4\alpha)\pi/4>0$, it follows that

$$\tilde g(t)>-B_1(f) \text{ on } [0,\ 1]. \tag{1.12}$$

On the other hand, on [0, 1],

$$\tilde g(t)-g(t)=4\alpha\left(\frac{5}{4}-t^2\right)\frac{\cos\pi t}{4t^2-1}<0$$

so that

$$\tilde g(t)<g(t)\leqslant B_1(f). \tag{1.13}$$

From (1.12) and (1.13) it follows that

(1.14) $$\max_{[0,1]} |\tilde{g}(t)| < B_1(f).$$

On $[1, \infty)$ we have, by (1.11) $\tilde{h}(t) \leqslant B_1(f) - \alpha$ and, by (1.14),

$$\tilde{h}(t) \geqslant \tilde{h}(1) = -\tilde{g}(1) > -B_1(f).$$

Hence on $[1, \infty)$ we have $\sup\limits_{[1,\infty)} |\tilde{h}(t)| < B_1(f)$ and so

(1.15) $$\sup_{[1,\infty)} |\tilde{g}(t)| < B_1(f).$$

Finally, from (1.14) and (1.15) we obtain

$$\sup_{[0,\infty)} |g(t)| < B_1(f),$$

a contradiction. Hence we must have $g(0) = -B_1(f)$.

To prove the alternation property (ii) suppose that $M = \max\limits_{[0,1]} g(t) < B_1(f)$. Put

$$\tilde{h}(t) = h(t) - \varepsilon\left(1 + \frac{2}{4t^2 - 1}\right) = f(t) - \left(x_0 + \varepsilon + \frac{x_1 + 2\varepsilon}{4t^2 - 1}\right)$$

and

$$\tilde{g}(t) = \tilde{h}(t) \cos \pi t,$$

where

$$0 < \varepsilon < \min\left(\frac{2}{5\pi}(B_1(f) - M),\ B_1(f)\right).$$

Since

$$\tilde{g}(t) = g(t) - \varepsilon\left(\frac{4t^2 + 1}{4t^2 - 1}\right) \cos \pi t,$$

we have, on $[0, 1]$,

$$-B_1(f) \leqslant g(t) < \tilde{g}(t) \leqslant M + \varepsilon \frac{5\pi}{2} < B_1(f)$$

so that

(1.16) $$\max_{[0,1]} |\tilde{g}(t)| < B_1(f).$$

Since $\tilde{h}$ is increasing on $[1, \infty)$, we have

$$\tilde{h}(t) \leqslant h(\infty) - \varepsilon \leqslant B_1(f) - \varepsilon$$

and, by the choice of ε,

$$\tilde{h}(t) \geqslant \tilde{h}(1) = -\tilde{g}(1) = -g(1) - \frac{5}{3}\varepsilon \geqslant -M - \frac{5}{3}\varepsilon > -B_1(f) + \varepsilon.$$

Hence

(1.17) $$\sup_{[1,\infty)}|\tilde{g}(t)|\leqslant\sup_{[1,\infty)}|\tilde{h}(t)|\leqslant B_1(f)-\varepsilon.$$

From (1.16) and (1.17) follows that $\sup_{[0,\infty)}|\tilde{g}(t)|<B_1(f)$, a contradiction which proves (ii).

In order to prove the last alternation property (iii), observe first that (iii) is equivalent to showing that

$$\limsup_{t\to\infty}|g(t)|=|f(\infty)-x_0|=B_1(f).$$

Since we always have $B_1(f)\geqslant|f(\infty)-x_0|$, it is sufficient to show that the hypothesis $B_1(f)>|f(\infty)-x_0|$ leads to a contradiction.

Suppose $B_1(f)>|f(\infty)-x_0|$. By (i) we have $h(0)=g(0)=-B_1(f)$ and $h(t)\to\infty$ as $t\to 1/2-0$. Since the function h is strictly increasing on [0, 1/2), it has exactly one zero at $t_1\in(0, 1/2)$. Put

$$\tilde{h}(t)=h(t)+\alpha\frac{t^2-t_1^2}{4t^2-1}$$

$$=f(t)-\left(x_0-\frac{\alpha}{4}+\frac{x_1-\alpha(1/4-t_1^2)}{4t^2-1}\right)$$

and $\tilde{g}(t)=\tilde{h}(t)\cos\pi t$. The number α is chosen so that

$$0<\alpha<\min(x_1/(1/4-t_1^2),\ 3(B_1(f)-|f(\infty)-x_0|)/2(1-t_1^2),$$
$$4B_1(f)/\pi(1/4-t_1^2)).$$

We shall show first that

(1.18) $$\max_{[0,1]}|\tilde{g}(t)|<B_1(f).$$

Since the function $\tilde{h}$ is increasing on [0, 1/2) and $h(0)=-B_1(f)$, we have

$$-B_1(f)<h(0)+\alpha t_1^2=\tilde{h}(0)\leqslant\tilde{h}(t)$$

and so

(1.19) $$\tilde{g}(t)>-B_1(f)\cos\pi t\geqslant -B_1(f) \text{ on } [0, 1/2).$$

On (1/2, ∞) we have

$$\tilde{h}(t)\leqslant\tilde{h}(\infty)=f(\infty)-x_0+\frac{\alpha}{4}.$$

Since $\frac{1}{4}\leqslant(1-t_1^2)/3$, we have

$$\tilde{h}(t)\leqslant|f(\infty)-x_0|+\alpha(1-t_1^2)/3.$$

In view of our choice of α we obtain finally the inequality

(1.20) $$\tilde{h}(t)\leqslant B_1(f)-\alpha(1-t_1^2)/3 \text{ on } (1/2, \infty).$$

From (1.20) follows, in particular, that

$$\check{g}(t) > B_1(f)\cos\pi t \geqslant -B_1(f) \text{ on } (1/2,\ 1]. \tag{1.21}$$

Next, to obtain the upper estimates for $\tilde{g}$, we observe that on $[0, t_1]$, $0<t_1<1/2$ we have $\tilde{h}(t) \leqslant \tilde{h}(t_1) = h(t_1) = 0$ and so

$$\tilde{g}(t) = \tilde{h}(t)\cos\pi t \leqslant 0. \tag{1.22}$$

On $[t_1, 1/2)$ or $(1/2, 1]$ we have $(\alpha(t^2-t_1^2)\cos\pi t)/(4t^2-1)<0$ and so

$$\tilde{g}(t) = g(t) + \alpha\frac{t^2-t_1^2}{4t^2-1}\cos\pi t < g(t) \leqslant B_1(f). \tag{1.23}$$

Finally, at $t=1/2$ we have $\tilde{g}(1/2) = g(1/2) - \alpha(1/4-t_1^2)\dfrac{\pi}{4}$.

Since $0<x_1\pi/4 = g(1/2) < B_1(f)$, we have, by the choice of α,

$$-B_1(f) < -\alpha(1/4-t_1^2)\frac{\pi}{4} < \tilde{g}(1/2) \leqslant B_1(f) - \tag{1.24}$$

$$-\alpha(1/4-t_1^2)\frac{\pi}{4} < B_1(f).$$

The inequalities (1.19) — (1.24) show that (1.18) holds true.

On $[1, \infty)$ we have, by the monotonicity of h,

$$\begin{aligned}\tilde{h}(t) \geqslant \tilde{h}(1) &= h(1) + \alpha(1-t_1^2)/3\\ &= -g(1) + \alpha(1-t_1^2)/3\\ &\geqslant -B_1(f) + \alpha(1-t_1^2)/3.\end{aligned}$$

This inequality and (1.20) show that

$$|\tilde{h}(t)| \leqslant B_1(f) - \alpha(1-t_1^2)/3 \text{ on } [1, \infty).$$

Hence

$$\sup_{[1,\infty)}|\tilde{g}(t)| \leqslant \sup_{[1,\infty)}|\tilde{h}(t)| \leqslant B_1(f) - \alpha(1-t_1^2)/3. \tag{1.25}$$

From (1.18) and (1.25) we obtain the desired contradiction.

To complete the proof of Theorem 1, we have only to show that the numbers x_0 and $x_1>0$ are uniquely determined.

Since $|f(\infty)-x_1| = B_1(f)$ and

$$-B_1(f) \leqslant -g(1) = h(1) < h(\infty) = f(\infty) - x_0 \leqslant |f(\infty)-x_0| = B_1(f),$$

we have $f(\infty)-x_0 = B_1(f)$ or $x_0 = f(\infty) - B_1(f)$. Since $g(0) = -B_1(f)$, we have $-B_1(f) = f(0) - x_0 + x_1$ and so $x_1 = f(\infty) - 2B_1(f)$.

2. An estimate from below for $B_1(f)$.

In this section we shall prove the following analog of a well known result of de la Vallée Poussin.

Theorem 2. *If there exists an error function*

$$g(t)=\left(f(t)-y_0-\frac{y_1}{4t^2-1}\right)\cos \pi t,$$

with $y_0 \leqslant f(\infty)$, *such that*

(i) $$g(0)=-\lambda_0<0$$

(ii) $$g(t_1)=\lambda_1>0 \text{ for some } t_1 \in (0, 1]$$

(iii) $$\limsup_{t\to\infty} |g(t)|=\lambda_\infty>0,$$

then

(2.1) $$B_1(f) \geqslant \min(\lambda_0, \lambda_1, \lambda_\infty).$$

Proof of Theorem 2 Let

$$g^*(t)=\left(f(t)-\overset{*}{x}_0-\frac{\overset{*}{x}_1}{4t^2-1}\right)\cos \pi t$$

be the extremal error function. Suppose that (2.1) is not true. We have then

(2.2) $$\sup_{[0,\infty)} |g^*(t)|=B_1(f)<\min(\lambda_0, \lambda_1, \lambda_\infty).$$

Put

$$\delta(t)=g^*(t)-g(t)=\left(y_0-\overset{*}{x}_0+\frac{y_1-\overset{*}{x}_1}{4t^2-1}\right)\cos \pi t.$$

In view of (i), (ii) and hypothesis (2.2), we have

$$\delta(0)=g^*(0)-g(0)=-B_1(f)+\lambda_0>0$$

and

$$\delta(t_1)=g^*(t_1)-g(t_1) \leqslant B_1(f)-\lambda_1<0.$$

Thus $\delta(c)=0$ for some $c \in (0, t_1)$, where $0<t_1 \leqslant 1$. Since $\cos \pi t/(4t^2-1)<0$ on $[0, 1]$, and

$$\delta(t)=\frac{\cos \pi t}{4t^2-1}\left((y_0-\overset{*}{x}_0)(4t^2-1)+y_1-\overset{*}{x}_1\right)$$

has a zero $c \in (0, 1]$, the polynomial $P(t)=(y_0-\overset{*}{x}_0)(4t^2-1)+y_1-\overset{*}{x}_1$ has zeros at $t=c$ and $t=-c$. Since $P(0)=-\delta(0)<0$, we must have $y_0-\overset{*}{x}_0>0$. It follows then, by (2.2)

$$f(\infty)-y_0<f(\infty)-\overset{*}{x}_0=B_1(f)<\lambda_\infty=|f(\infty)-y_0|,$$

or $f(\infty)-y_0<|f(\infty)-y_0|$. Hence we have found that $y_0>f(\infty)$, contrary to our hypothesis $y_0 \leqslant f(\infty)$. This completes the proof of Theorem 2.

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