PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 18 (32), 1975, pp. 157—164

ON PSEUDO METRIC SEMI-SYMMETRIC CONNECTIONS

M. Prvanović

(Communicated October 15, 1974)

1. Introduction. — Let M be an n-dimensional differentiable manifold of class C^∞. Let there be given a Riemannian metric g in M, and $\overset{0}{\nabla}$ denote the Levi-Civita connection with recpect to the Riemannian metric g. If π is a 1-form, let P be a vector field defined by

$$g(X, P) = \pi(X)$$

for any vector field X.

Let ∇ be an other linear connection in M. When ∇ satisfies

$$\nabla_X Y = \overset{0}{\nabla}_X Y + \pi(Y) X - g(X, Y) P$$

for any vector fields X, Y, a linear connection ∇ is called semi-symmetric metric connection [1], [2]. This linear connection has also appeared in [3]. Following the notations in [3] we shall indicate such linear connection by $\overset{2}{\nabla}$, that is

$$\overset{2}{\nabla}_X Y = \overset{0}{\nabla}_X Y + \pi(Y) X - g(X, Y) P. \tag{1.1}$$

For the covariant differentiation of a 1-form ω, we have

$$(\overset{2}{\nabla}_X \omega)(Y) = (\overset{0}{\nabla}_X \omega)(Y) - \omega(X)\pi(Y) + \omega(P) g(X, Y).$$

The connection (1.1) being given, we can consider the connection

$$\begin{cases} \overset{1}{\nabla}_X Y = \overset{0}{\nabla}_X Y + \pi(X) Y - g(X, Y) P \\ (\overset{1}{\nabla}_X \omega)(Y) = (\overset{0}{\nabla}_X \omega)(Y) - \omega(Y)\pi(X) + \omega(P) g(X, Y) \end{cases} \tag{1.2}$$

as well as the connections

$$\overset{3}{\nabla}(\omega Y) = \omega(\overset{1}{\nabla} Y) + (\overset{2}{\nabla}\omega)(Y) \tag{1.3}$$

$$\overset{4}{\nabla}(\omega Y) = \omega(\overset{2}{\nabla} Y) + (\overset{1}{\nabla}\omega)(Y). \tag{1.4}$$

If $g=(g_{ij})$ be the Riemannian metric on M with respect to coordinates $(x^1, x^2, \ldots, x^n)$, and g^{ij} are the components of the corresponding tensor field of type (2,0), we have

$$\overset{2}{\nabla}_k g_{ij}=\overset{2}{\nabla}_k g^{ij}=0,$$

i.e. the connection $\overset{2}{\nabla}$ is a metric connection.

The other three connections are not metric. Indeed, it is easy to see that

$$\overset{3}{\nabla}_k g_{ij}=0, \qquad \overset{4}{\nabla}_k g^{ij}=0, \tag{1.5}$$

but

$$\overset{1}{\nabla}_k g_{ij}=\overset{4}{\nabla}_k g_{ij}=-2\, P_k g_{ij}+P_i g_{jk}+P_j g_{ik} \tag{1.6}$$

and

$$\overset{1}{\nabla}_k g^{ij}=\overset{3}{\nabla}_k g^{ij}=2\, P_k g^{ij}-P^i \delta^j_k-P^j \delta^j_k \tag{1.7}$$

where P_i and P^i are the components of the form π and of the vector field P with respect to local coordinates.

Taking into consideration (1.5), (1.6) and (1.7), we call the connection $\overset{3}{\nabla}$ and $\overset{4}{\nabla}$ the pseudo-metric semi-symmetric connections.

The purpose of the present paper is to study some properties of connections $\overset{3}{\nabla}$ and $\overset{4}{\nabla}$ in a Riemannian manifold. We consider, in § 2, a Riemannian manifold which admits a connection $\overset{3}{\nabla}$ whose curvature tensor and Ricci tensor vanish, and in § 3 the same problems concerning a connection $\overset{4}{\nabla}$.

2. Curvature tensor of the connection $\overset{3}{\nabla}$. — Following the notation in [3], we shall denote by $\underset{3}{R}{}^i_{rkj}$ the components of the curvature tensor of the connection $\overset{3}{\nabla}$ with respect to coordinates $(x^1, x^2, \ldots, x^n)$, that is

$$\overset{2}{\nabla}_k \overset{1}{\nabla}_j z^i-\overset{1}{\nabla}_j \overset{2}{\nabla}_k z^i=\underset{3}{R}{}^i_{ajk}\, z^a,$$

where z^i are the components of vector field Z. Consequently we can put

$$\underset{3}{R}(X, Y)Z=\overset{2}{\nabla}_X \overset{1}{\nabla}_Y Z-\overset{1}{\nabla}_Y \overset{2}{\nabla}_X Z+\overset{2}{\nabla}_{\overset{1}{\nabla}_Y X} Z-\overset{1}{\nabla}_{\overset{2}{\nabla}_X Y} Z. \tag{2.1}$$

Substituting (1.1) and (1.2) into (2.1), we find

$$\underset{3}{R}(X, Y)Z=\overset{2}{\nabla}_X(\overset{0}{\nabla}_Y Z+\pi(Y)Z-g(Y, Z)P)-\overset{1}{\nabla}_Y(\overset{0}{\nabla}_X Z+\pi(Z)X-g(X, Z)P)$$
$$+\overset{0}{\nabla}_{\overset{1}{\nabla}_Y X} Z+\pi(Z)\overset{1}{\nabla}_Y X-g(\overset{1}{\nabla}_Y X, Z)\, P$$

$$-\overset{0}{\nabla}_{\overset{2}{\nabla}_X Y} Z-\pi(\overset{2}{\nabla}_X Y)Z+g(\overset{2}{\nabla}_X Y, Z)P$$

$$=\overset{0}{\nabla}_X(\overset{0}{\nabla}_Y Z+\pi(Y)Z-g(Y, Z)P)+\pi(\overset{0}{\nabla}_Y Z+\pi(Y)Z-g(Y, Z)P)X$$

$$-g(X, \overset{0}{\nabla}_Y Z+\pi(Y)Z-g(Y, Z)P)P$$

$$-\overset{0}{\nabla}_Y(\overset{0}{\nabla}_X Z+\pi(Z)X-g(X, Z)P)-\pi(Y)(\overset{0}{\nabla}_X Z+\pi(Z)X-g(X, Z)P)$$

$$+g(Y, \overset{0}{\nabla}_X Z+\pi(Z)X-g(X, Z)P)P$$

$$+(\overset{0}{\nabla}_{\overset{1}{\nabla}_Y X} Z+\pi(Z)\overset{1}{\nabla}_Y X-g(\overset{1}{\nabla}_Y X, Z)P)$$

$$-(\overset{0}{\nabla}_{\overset{2}{\nabla}_X Y} Z+\pi(\overset{2}{\nabla}_X Y)Z-g(\overset{2}{\nabla}_X Y, Z)P),$$

from which

$$\underset{3}{R}(X, Y)Z=K(X, Y)Z+\{(\overset{0}{\nabla}_X \pi)(Y)-\pi(Y)\pi(X)+g(X, Y)\pi(P)\}Z$$

$$-\{(\overset{0}{\nabla}_Y \pi)(Z)X-\pi(Z)\pi(Y)X+g(Y, Z)\pi(P)\}X$$

$$+g(X, Z)\{\overset{0}{\nabla}_Y P-g(Y, P)P+Y\pi(P)\}$$

$$-g(Y, Z)\{\overset{0}{\nabla}_X P-g(X, P)P+X\pi(P)\}$$

$$-g(X, Z)\pi(P)Y+g(Y, Z)\pi(P)X,$$

that is

$$\underset{3}{R}(X, Y)Z=K(X, Y)Z+\beta(X, Y)Z-\beta(Y, Z)X+g(X, Z)BY-g(Y, Z)BX$$

(2.2)

$$-g(X, Z)\pi(P)Y+g(Y, Z)\pi(P)X,$$

where

$$K(X, Y)Z=\overset{0}{\nabla}_X\overset{0}{\nabla}_Y Z-\overset{0}{\nabla}_Y\overset{0}{\nabla}_X Z-\overset{0}{\nabla}_{[X, Y]}Z$$

is the curvature tensor of the connection $\overset{0}{\nabla}$, β is a tensor field of type (0,2) defined by

$$\beta(X, Y)=(\overset{0}{\nabla}_X \pi)(Y)-\pi(Y)\pi(X)+g(X, Y)\pi(P)$$

and B is a tensor field of type (1,1) defined by

$$g(BX, Y)=\beta(X, Y), \tag{2.3}$$

for any vector fields X and Y.

We now put:

$$K(X, Y, Z, W)=g(K(X, Y)Z, W), \qquad \underset{3}{R}(X, Y, Z, W)=g(\underset{3}{R}(X, Y)Z, W)$$

$$S(Y, Z)=\sum_i K(X_i, Y, Z, X_i), \qquad r=\sum_i S(X_i, X_i)$$

$$'\underset{3}{\rho}(Y, Z)=\sum_i \underset{3}{R}(X_i, Y, Z, X_i), \qquad '\underset{3}{R}=\sum_i {}'\underset{3}{\rho}(X_i, X_i)$$

$$''\underset{3}{\rho}(Y, W)=\sum_i \underset{3}{R}(X_i, Y, X_i, W), \qquad ''\underset{3}{R}=\sum_i {}''\underset{3}{\rho}(X_i, X_i)$$

$$'''\underset{3}{\rho}(Z, W)=\sum_i \underset{3}{R}(X_i, X_i, Z, W), \qquad '''\underset{3}{R}=\sum_i {}'''\underset{3}{\rho}(X_i, X_i)$$

X, Y, Z, W being arbitrary vector fields, and X_i $(i=1, 2, \ldots, n)$ being n orthonormal vectors. Then we have from (2.2)

$$\beta(X, Y)=\frac{1}{n-1}\,'''\underset{3}{\rho}(X, Y) \quad \text{and} \quad b=\frac{1}{n-1}\,'''\underset{3}{R}, \tag{2.4}$$

where b is trace of B.

We get, too

$$'\underset{3}{\rho}(X, Z)=g(X, Z)\,b-\beta(X, Z)-(n-1)\,g(X, Z)\,\pi(P)+S(X, Z)$$

from which

$$'\underset{3}{R}=(n-1)\,b-(n-1)\,n\,\pi(P)+r,$$

and consequently, taking into account (2.4),

$$\pi(P)=-\frac{1}{n(n-1)}('\underset{3}{R}-{}'''\underset{3}{R}-r). \tag{2.5}$$

Substituting (2.4) and (2.5) in (2.2), we find

$$\underset{3}{R}(X, Y)Z-\frac{1}{n-1}[\,'''\underset{3}{\rho}(X, Y)Z-{}'''\underset{3}{\rho}(Y, Z)X+g(X, Z)\,'''\underset{3}{R}(Y)-g(Y, Z)\,'''\underset{3}{R}(X)]$$

$$+\frac{1}{n(n-1)}('\underset{3}{R}-{}'''\underset{3}{R})[g(Y, Z)X-g(X, Z)Y]$$

$$=K(X, Y)Z+\frac{r}{n(n-1)}[g(Y, Z)X-g(X, Z)Y],$$

where the tensor field $\underset{3}{R}(X)$ of type (1.1) is defined by

$$g('''\underset{3}{R}(X), Y)={}'''\underset{3}{\rho}(X, Y).$$

The tensor on the right of this equation being the concircular curvature tensor [4], we have the theorems:

The tensor

$$\underset{3}{R}(X, Y)Z-\frac{1}{n-1}[{}'''\underset{3}{\rho}(X, Y)Z-{}'''\underset{3}{\rho}(Y, Z)X$$

$$+g(X, Z)\,{}'''\underset{3}{R}(Y)-g(Y, Z)\,{}'''\underset{z}{R}(X)]$$

$$+\frac{1}{n(n-1)}({}'\underset{3}{R}-{}'''\underset{3}{R})[g(Y, Z)X-g(X, Z)Y]$$

is independent of a 1*-form* π. *It is equal to the concircular curvature tensor of the Riemannian manifold M.*

If the Riemannian manifold admits a pseudo-metric semi-symmetric connection whose curvature tensor $\underset{3}{R}(X, Y)Z$ *vanishes, the Riemannian metric is concirculary flat.*

If the Riemannian manifold admits a pseudo-metric semi-symmetric connection whose Ricci tensor ${}'''\underset{3}{\rho}(X, Y)$ *and curvature scalar* ${}'\underset{3}{R}$ *vanish, then the curvature tensor* $\underset{3}{R}(X, Y)Z$ *of the connection* $\overset{3}{\nabla}$ *is equal to the concircular curvature tensor of the Riemannian manifold.*

3. Curvature tensor of the connection $\overset{4}{\nabla}$. — Let us consider the tensor

$$\underset{4}{R}(X, Y)Z=\overset{2}{\nabla}_X\overset{1}{\nabla}_Y Z-\overset{1}{\nabla}_Y\overset{2}{\nabla}_X Z+\overset{2}{\nabla}_{\overset{2}{\nabla}_Y X} Z-\overset{1}{\nabla}_{\overset{1}{\nabla}_X Y} Z. \tag{3.1}$$

Substituting (1.1) and (1.2) into (3.1), we find

$$\underset{4}{R}(X, Y)Z=\overset{0}{\nabla}_X(\overset{1}{\nabla}_Y Z)+\pi(\overset{1}{\nabla}_Y Z)X-g(X, \overset{1}{\nabla}_Y Z)P$$

$$-\overset{0}{\nabla}_Y(\overset{2}{\nabla}_X Z)-\pi(Y)\overset{2}{\nabla}_X Z+g(Y, \overset{2}{\nabla}_X Z)P$$

$$+\overset{0}{\nabla}_{\overset{2}{\nabla}_Y X} Z+\pi(Z)\overset{2}{\nabla}_Y X-g(\overset{2}{\nabla}_Y X, Z)P$$

$$-\overset{0}{\nabla}_{\overset{1}{\nabla}_X Y} Z-\pi(\overset{1}{\nabla}_X Y)Z+g(\overset{1}{\nabla}_X Y, Z)P$$

$$=\overset{0}{\nabla}_X(\overset{0}{\nabla}_Y Z+\pi(Y)Z-g(Y, Z)P)+\pi(\overset{0}{\nabla}_Y Z+\pi(Y)Z-g(Y, Z)P)X$$

$$-g(X, \overset{0}{\nabla}_Y Z+\pi(Y)Z-g(Y, Z)P)P$$

$$-\overset{0}{\nabla}_Y(\overset{0}{\nabla}_X Z+\pi(Z)X-g(X, Z)P)-\pi(Y)(\overset{0}{\nabla}_X Z+\pi(Z)X-g(X, Z)P)$$

$$+g(Y, \overset{0}{\nabla}_X Z+\pi(Z)X-g(X, Z)P)P$$

$$-\overset{0}{\nabla}_{(\overset{0}{\nabla}_Y X+\pi(X)Y-g(X,Y)P)} Z+\pi(z)(\overset{0}{\nabla}_Y X+\pi(X)Y-g(X,Y)P)$$

$$-g(Z,\overset{0}{\nabla}_Y X+\pi(X)Y-g(X,Y)P)P$$

$$-\overset{0}{\nabla}_{(\overset{0}{\nabla}_X Y+\pi(X)Y-g(X,Y)P)} Z-\pi(\overset{0}{\nabla}_X Y+\pi(X)Y-g(X,Y)P)Z$$

$$+g(Z,\overset{0}{\nabla}_X Y+\pi(X)Y-g(X,Y)P)P,$$

from which:

$$\underset{4}{R}(X,Y)Z=K(X,Y)Z+(\overset{0}{\nabla}_X\pi)(Y)Z-(\overset{0}{\nabla}_Y\pi)(Z)X+g(X,Z)\overset{0}{\nabla}_Y P-$$

$$-g(Y,Z)\overset{0}{\nabla}_X P$$

$$+g(Y,Z)[g(X,P)P-\pi(P)X]-g(X,Z)[\pi(Y)P-\pi(P)Y]$$

$$+[\pi(X)\pi(Z)-\pi(P)g(X,Z)]Y-[\pi(X)\pi(Y)-g(X,Y)\pi(P)]Z.$$

If we put

$$(\overset{0}{\nabla}_X\pi)(Y)=\gamma(X,Y) \quad \text{and} \quad \pi(X)\pi(Y)-g(X,Y)\pi(P)=\delta(X,Y),$$

we obtain:

$$\underset{4}{R}(X,Y)Z=K(X,Y)Z+\gamma(X,Y)Z-\gamma(Y,Z)X+g(X,Z)G(Y)-g(Y,Z)G(X)$$

(3.2)
$$-\delta(X,Y)Z+\delta(X,Z)Y-g(X,Z)D(Y)+g(Y,Z)D(X),$$

where G and D are the tensor fields of type (1.1) defined by

$$g(GX,Y)=\gamma(X,Y), \qquad g(DX,Y)=\delta(X,Y)$$

respectively.

Now, we put

$$\underset{4}{R}(X,Y,Z,W)=g(\underset{4}{R}(X,Y)Z,W)$$

$$'\underset{4}{\rho}(Y,Z)=\sum_i \underset{4}{R}(X_i,Y,Z,X_i), \qquad '\underset{4}{R}=\sum_i {}'\underset{4}{\rho}(X_i,X_i)$$

$$''\underset{4}{\rho}(Y,W)=\sum_i \underset{4}{R}(X_i,Y,X_i,W), \qquad ''\underset{4}{R}=\sum_i {}''\underset{4}{\rho}(X_i,X_i)$$

$$'''\underset{4}{\rho}(Z,W)=\sum_i \underset{4}{R}(X_i,X_i,Z,W), \qquad '''\underset{4}{R}=\sum_i {}'''\underset{4}{\rho}(X_i,X_i)$$

X, Y, Z, W being arbitrary vector fields, and X_i $(i=1, 2, \ldots, n)$ being n orthonormal vectors. Then we obtain from (3.2)

(3.3) $$'\underset{4}{\rho}(X,Z)=S(X,Z)+n\,\delta(X,Z)-\gamma(X,Z)+g(X,Z)(G-D),$$

(3.4) $$''\underset{4}{\rho}(X,Y)=-S(Z,Y)+\gamma(Z,Y)-(n-1)\gamma(Y,Z)-\delta(Y,Z)$$
$$-g(Y,Z)(G-D),$$

(3.5) $$'''\underset{4}{\rho}(X,Y)=(n-1)[\gamma(X,Y)-\delta(X,Y)].$$

We get from (3.3)

$$G=\frac{1}{n-1}({}'\underset{4}{R}-r). \tag{3.6}$$

It is easy to see that from (3.3) and (3.5) follow the equations

$$\delta(X,Y)=\frac{1}{n-1}\left[{}'\underset{4}{\rho}(X,Y)+\frac{1}{n-1}\,{}'''\underset{4}{\rho}(X,Y)-S(X,Y)-g(X,Y)(G-D)\right] \tag{3.7}$$

$$\gamma(X,Y)=\frac{1}{n-1}\left[{}'\underset{4}{\rho}(X,Y)+\frac{n}{n-1}\,{}'''\underset{4}{\rho}(X,Y)-S(X,Y)-g(X,Y)(G-D)\right], \tag{3.8}$$

and from (3.4), the equation

$${}''\underset{4}{R}=-r+(1-n)(2G-D).$$

Substituting G from (3.6), we get

$$D=\frac{1}{n-1}({}''\underset{4}{R}+2\,{}'\underset{4}{R}-r).$$

Consequently

$$G-D=-\frac{1}{n-1}({}'\underset{4}{R}+{}''\underset{4}{R})$$

or, putting $\underset{4}{R}=-({}'\underset{4}{R}+{}''\underset{4}{R})$,

$$G-D=\frac{1}{n-1}\underset{4}{R}.$$

Substituting this in (3.7) and (3.8), and then (3.7) and (3.8) into (3.2), we obtain:

$$\begin{aligned}
&\underset{4}{R}(X,Y)Z-{}'''\underset{4}{\rho}(X,Y)Z-g(X,Z)\,{}'''\underset{4}{R}(Y)+g(Y,Z)\,{}'''\underset{4}{R}(X)\\
&+\frac{1}{n-1}\left[{}'\underset{4}{\rho}(Y,Z)+\frac{n}{n-1}\,{}'''\underset{4}{\rho}(Y,Z)-\frac{1}{n-1}g(Y,Z)\underset{4}{R}\right]X\\
&-\frac{1}{n-1}\left[{}'\underset{4}{\rho}(X,Z)+\frac{1}{n-1}\,{}'''\underset{4}{\rho}(X,Z)-\frac{1}{n-1}g(X,Z)\underset{x}{R}\right]Y\\
&=K(X,Y)Z+\frac{1}{n-1}[S(Y,Z)X-S(X,Z)Y],
\end{aligned} \tag{3.9}$$

where the tensor field ${}'''\underset{4}{R}(X)$ is defined by

$$g({}'''\underset{4}{R}(X),Y)={}'''\underset{4}{R}(X,Y).$$

The tensor on the right of (3.9) being the projective curvature tensor $P(X,Y)Z$ of the connection $\overset{0}{\nabla}$, we have the theorem:

The tensor on the left of (3.9) is independent of a 1*-form* π*. It is equal to the projective curvature tensor of the Riemannian manifold.*

Let $\underset{4}{R}(X, Y)Z=0$. Then the projective curvature tensor of the connection $\overset{0}{\nabla}$ vanishes. Conversely, if the Riemannian manifold M is projectively flat, then equations

$$(\overset{0}{\nabla}_X \pi)(Y)=\pi(X)\pi(Y)-g(X, Y)\pi(P)=\frac{1}{1-n}\,S(X, Y)$$

are completely integrable. Consequently a pseudo-metric connection $\overset{4}{\nabla}$ whose curvature tensor $\underset{4}{R}(X, Y)Z$ vanishes, exist. Thus we get the theorem:

In order that a Riemannian metric admits a pseudo-metric connection $\overset{4}{\nabla}$ *whose curvature tensor vanishes, it is necessary and sufficient that the Riemannian metric be projectively flat.*

Also we have the theorem:

If the Riemannian manifold admits a pseudo-metric connection whose Ricci tensors $'''\underset{4}{\rho}(X, Y)$ *and* $'\underset{4}{\rho}(X, Y)$ *vanish, then the curvature tensor* $\underset{4}{R}(X, Y)Z$ *of the connection* $\overset{4}{\nabla}$ *is equal to the projective curvature tensor of the Riemannian manifold.*

REFERENCES

[1] K. Yano, *On semi-symmetric metric connection*, Revue Roumaine de Mathématiques pures et appliquées, 15 (1970), 1579—1986.

[2] T. Imai, *Notes on semi-symmetric metric connections*, Tensor, N. S. vol. 24 (1972), 293—296.

[3] M. Prvanović, *Some tensors of metric semi-symmetric connection*, Atti della Accad. delle Sci. di Torino, vol 107 (1972—1973) 303—316.

[4] K. Yano, *Concircular geometry I, II, III, IV*, Proc. Imper. Acad. Tokyo, 16 (1940) 195—200, 354—360, 442—448, 505—511.