PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle, série tome 18 (32), 1975, pp. 149—155

TWO INTEGRAL EQUATIONS IN THE FIELD OF MIKUSIŃSKI OPERATORS

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(Communicated June 27, 1974)

Introduction

Let S be the subset of the field M of Mikusiński operators whose elements have the following property: In the class of equivalence which defines the element $a \in M$ exist such elements $f, g \in \mathscr{C}$ ($\mathscr{C}$ is the set of continuous complex-valued functions of non-negative real variable), $g \not\equiv 0$, $a = \frac{f}{g}$, which have the absolute convergent Laplace transformations in the halfplane $\operatorname{Re} z \geqslant x_0$, where $x_0 > 0$ and x_0 changes according to f and g. With the induced operations from M, S becomes the field and it is known that $S \neq M$ [3].

If we denote Laplace transformation of the function f as $\mathscr{L}\{f\}$ then

$$a = \frac{f}{g} \in S \quad \mathscr{L}\{a\} = \frac{\mathscr{L}\{f\}}{\mathscr{L}\{g\}}.$$

Let

$$\hat{S} = \{\mathscr{L}\{a\} : a \in S\}$$

then $(\hat{S}, +, \cdot)$ is also the field.

Theorem D. There exist an algebraic isomorphism between S and $\hat{S}$ [3]

$$S \underset{\mathcal{L}^{-1}}{\overset{\mathcal{L}}{\rightleftarrows}} \hat{S}\cdot$$

Ditkin's result is very important because it connects the operators with Laplace transformations which have been well studied. Following the ideas of Erdélyi [5] and from this isomorphism we have

$a \in S$	$\mathcal{L}\{a\} \in \hat{S}$
$l^{\alpha} = s^{-\alpha} = \left\{\frac{t^{\alpha-1}}{\Gamma(\alpha)}\right\}$	$z^{-\alpha}$
$\frac{1}{(s-\lambda)^n} = \left\{\frac{t^{n-1}}{(n-1)!} e^{\lambda t}\right\}$	$\frac{1}{(z-\lambda)^n}$
$\frac{s^{c-d}}{(s-\lambda)^c} = \left\{\frac{t^{d-1}}{\Gamma(d)} \, {}_1F_1(c; d: \lambda t)\right\}$	$\frac{z^{c-d}}{(z-\lambda)^c}$

where s is the differential operator in the field M, l is the integral operator in M and ${}_1F_1(c, d, z)$ denotes the confluent hypergeometric function with $\operatorname{Re} d > 0$. As usual ${}_1F_1(a, b, z)$ is defined by the series

$$ {}_1F_1(a, b, z) = \sum_{n=\infty}^{\infty} \frac{(a)_n z^n}{(b)_n n!} \qquad (a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} $$

for all complex values of the parameters a and b exept $b = 0, -1, -2, \ldots$ The series is absolutely convergent for all finite z so that ${}_1F_1(a, b, z)$ is an entire function of z. If $a = b$ then ${}_1F_1(a, a, z) = e^z$. Recently T. R. Prabhaker [7] has used fractional integrals in order to obtain explicit solutions to a convolution integral equation in which the kernel involved a confluent hypergeometric function. Following ideas of Erdélyi [5] this equation

$$ \int_0^t K(a, b, \lambda(t-u)) F(u)\, du = g(t) \tag{1} $$

in which the kernel is of the form

$$ K(a, b, \lambda t) = \frac{t^{b-1}}{\Gamma(b)}\, {}_1F_1(a, b, \lambda t) \qquad \operatorname{Re} b > 0 $$

can be studied from the standpoint of Mikusiński operators. Bushman [2] solved the integral equation (1) using Mikusiński operators.

We investigate the conditions under which a locally integrable solution of two convolution integral equations exists. If we denote with $\Phi(\beta, -\nu; z)$ the Wright function which is defined by

$$ \Phi(\beta, -\nu; z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n+1)\Gamma(\beta - n\nu)} \qquad 0 < \nu < 1 $$

then the first equation is

$$ \int_0^t K_\nu(a, b, \lambda, t-u) f(u)\, du = g(t) \tag{2} $$

in which the kernel is of the form

$$ K_\nu(a, b, \lambda, t) = \int_0^\infty t^{-1} \Phi(0, -\nu; -ut^{-\nu}) K(a, b, \lambda u)\, du $$

and the second integral equation is

$$ \int_0^t K_{\nu\mu}(a, b, \lambda, t-u) f(u)\, du = g(t) \tag{3} $$

in which the kernel is of the form

$$ K_{\nu\mu}(a, b, \lambda, t) = \int_\mu^\infty t^{-1} \Phi(0, -\nu; -ut^{-\nu}) K(a; b; \lambda(u-\mu))\, du $$

These equations we will investigate from the standpoint of Mikusiński calculus.

The general field of electromagnetic waves in spherically stratified isotropic media consists of a superposition of two fields known as the field of electric type and the field of magnetic type. The field of electric type satisfies one differential equation which possesses exact solutions expressible in terms of hypergeometric, Whittaker or Bessel functions. We think the equations (2) and (3) will be interesting in the theory of electromagnetic wave propagation.

The representation two operators in M

Lemma 1. *If $t>0$, $0<\nu<1$, a and b are complex numbers with $\operatorname{Re} b>0$, then the operator*

$$\frac{s^{(a-b)\nu}}{(s^\nu-\lambda)^a}$$

can be written in the field M in the following form:

$$\frac{s^{(a-b)\nu}}{(s^\nu-\lambda)^a}=sl^\nu\left\{\int_0^\infty \Phi(0,\ -\nu;\ -ut^{-\nu})\,K(a;\ b;\ \lambda u)\frac{du}{\lambda u}\right\} \tag{2.1}$$

$$=s\left\{\int_0^\infty \Phi(1,\ -\nu;\ -ut^{-\nu})\,K(a;\ b;\ \lambda u)\,du\right\}$$

$$=\left\{\int_0^\infty t^{-1}\,\Phi(0,\ -\nu;\ -ut^{-\nu})\,K(a;\ b;\ \lambda u)\,du\right\}.$$

Proof. Using the following known relation [8]. If $f\in L$ (for the set L see [4]) and $\mathcal{L}\{f\}=F(z)$ then for $0<\nu<1$ is

$$\mathcal{L}\left\{\int_0^\infty \Phi(0,\ -\nu;\ -tx^{-\nu})\,f(t)\frac{du}{\nu t}\right\}=z^{\nu-1}\,F(z^\nu). \tag{4}$$

Let us consider the function $F(z)=\dfrac{z^{a-b}}{(z-\lambda)^a}=\mathcal{L}\{K(a,\,b,\,\lambda t\}$ $\operatorname{Re} b>0$. Let $\mathcal{L}$ denote the linear space of (equivalent classes of) complex-valued functions $f(t)$ which are Lebesque-integrable on $[0,\,T]$, $T<\infty$.

The function $K(a,\,b,\,t)$ has the following property:

For $\operatorname{Re} b>0$, $t^{b-1}\,{}_1F_1(a;\,b;\,t)\in\mathcal{L}$, ${}_1F_1(a,\,b,\,t)$ being an entire function. The entire function ${}_1F_1(a,\,b,\,u)$ is bounded in $[0,\,T]$, let

$$|{}_1F_1(a,\,b,\,u)|\leqslant M,\ u\in[0,\,T]$$

so that

$$\int_0^T |u^{b-1}\,{}_1F_1(a,\,b,\,u)|\,du\leqslant M\int_0^T u^{\operatorname{Re} b-1}\,du$$

which is finite for $\operatorname{Re} b>0$.

From this proprety follows

$$K(a, b, \lambda t)=\left[\frac{t^{b-1}}{\Gamma(b)}\,{}_1F_1(a; b; \lambda t)\right]\in L \text{ for } \operatorname{Re} b>0$$

and we apply for the function $K(a, b, \lambda t)$ the relation (4). According to (4) for $0<\nu<1$ values:

$$\mathscr{L}\left\{\int_0^\infty \Phi(0, -\nu, -xt^{-\nu})\,K(a; b; \lambda x)\frac{dx}{\nu x}\right\}=z^{\nu-1}\frac{z^{(a-b)\nu}}{(z^\nu-\lambda)^a} \quad \operatorname{Re} b>0.$$

From the theorem D and the last relation follows

(2.1-1) $$s^{\nu-1}\frac{s^{(a-b)\nu}}{(s^\nu+\lambda)^a}=\left\{\int_0^\infty \Phi(0, -\nu, -xt^{-\nu})\,K(a, b, -\lambda x)\frac{dx}{\nu x}\right\}.$$

The integral

$$\psi(t)=\int_0^\infty \Phi(0, -\nu; -xt^{-\nu})\,K(a; b; \lambda x)\frac{dx}{\nu x}$$

exists. If follows from the following properties of the function ${}_1F_1$:
(A) If $\lambda>0$ and $x\to\infty$ or $x\lambda\to\infty$ then

$${}_1F_1(a; b; \lambda x)=\frac{\Gamma(b)}{\Gamma(a)}e^{\lambda x}(\lambda x)^{a-b}[1+0(|\lambda x|^{-1})].$$

If $\lambda<0$ and $x\to\infty$ or $\lambda x\to-\infty$ then

$${}_1F_1(a, b, \lambda x)=\frac{\Gamma(b)}{\Gamma(b-a)}(-\lambda x)^{-a}[1+0(|\lambda x|^{-1})].$$

(B) Taylor gave the asymptotic behaviour ${}_1F_1$ in the uniform neighbourhood of the point zero [1] (p. 268). Let $\varkappa=b/2-a$ and $|\arg\varkappa|<\pi$ and $\varkappa\lambda x$ is bounded then

$${}_1F_1(a; b; \lambda x)=\Gamma(b)(\varkappa\lambda x)^{1/2-b/2}e^{\lambda x/2}J_{b-1}\left(2\sqrt{\varkappa\lambda x}\right)+0(|\varkappa|^{-1})$$

where $J_{b-1}(x)$ is Bessel function.

And the following properties of Wright function

(C) $$|\Phi(0, -\nu; -xt^{-\nu})\,t^\nu/x|\leqslant A(\nu), \quad \frac{t}{x^{1/\nu}}\geqslant 0$$

(D) $$\Phi(0, -\nu; -xt^{-\nu})\sim\sqrt{a}\,t^{-\frac{\nu}{2(1-\nu)}}x^{\frac{1}{2(1-\nu)}}\exp\left(at^{-\frac{\nu}{1-\nu}}x^{\frac{1}{1-\nu}}\right)$$

$$x\to\infty, \quad a=(1-\nu)^{\frac{\nu}{1-\nu}}.$$

Besides (*C*) and (*D*) Wright function has also the properties

(*E*) $$l^{\nu}\left\{\Phi(0, -\nu; -xt^{-\nu})\frac{1}{\nu x}\right\}=\{\Phi(1, -\nu; -xt^{-\nu})\}$$

(*F*) $$s\{\Phi(1, -\nu; -xt^{-1})\}=\{t^{-\nu}\Phi(0, -\nu; -xt^{-\nu})\}.$$

Namely,

$$\frac{s^{\nu(a-b)}}{(s^{\nu}-\lambda)^{a}}=sl^{\nu}\left\{\int_{0}^{\infty}\Phi(0, -\nu; -xt^{-\nu})K(a; b; \lambda x)\frac{dx}{\nu x}\right\}$$

$$=s\left\{\int_{0}^{\infty}\frac{(t-u)^{\nu-1}}{\Gamma(\nu)}du\frac{dx}{\nu x}\int_{0}^{\infty}\Phi(0, -\nu; -xu^{-\nu})K(a; b; \lambda x)\frac{dx}{\nu x}\right\}.$$

From the properties (*A*)—(*D*) we may interchange the order of integration and then according to the properties (*E*) and (*F*) we have:

$$\frac{s^{(a-b)\nu}}{(s^{\nu}-\lambda)^{a}}=s\left\{\int_{0}^{\infty}K(a; b; \lambda x)\frac{dx}{\nu x}\int_{0}^{t}\frac{(t-u)^{\nu-1}}{\Gamma(\nu)}\Phi(0, -\nu; -xu^{-\nu})du\right\}$$

$$=s\left\{\int_{0}^{\infty}K(a: b: \lambda x)\Phi(1, -\nu; -xt^{-\nu})dx\right\}$$

$$=\left\{\int_{0}^{\infty}t^{-1}\Phi(0, -\nu; -xt^{-\nu})K(a; b; \lambda x)dx\right\}.$$

Lemma 2. *If* $0<\nu<1$, $\lambda\in R$, $\mu>0$ *and* a *and* b *are complex numbers with* $\mathrm{Re}\,b>0$, *then the operator*

$$\frac{s^{\nu(a-b)}}{(s^{\nu}-\lambda)^{a}}\cdot e^{-\mu s^{\nu}}$$

has in the field M, the following representation

(3.1) $$\frac{s^{(a-b)\nu}}{(s^{\nu}-\lambda)^{a}}e^{-\mu s^{\nu}}=sl^{\nu}\left\{\int_{\mu}^{\infty}\Phi(0, -\nu; -xt^{-\nu})K(a; b; \lambda(x-\mu))\frac{dx}{\nu x}\right\}$$

$$=s\left\{\int_{\mu}^{\infty}\Phi(1, -\nu; -xt^{-\nu})K(a; b; \lambda(x-\mu))dx\right\}$$

$$=\left\{\int_{\mu}^{\infty}t^{-1}\Phi(0, -\nu; -xt^{-\nu})K(a, b, \lambda(x-\mu))dx\right\}.$$

Proof. $K(a, b, \lambda t)\in\mathscr{L}$ and then is

$$e^{-\mu s}K(a; b; \lambda t)=\begin{Bmatrix}0 & 0\leqslant t<\mu\\ K(a; b; \lambda(t-\mu)), & t\geqslant\mu\end{Bmatrix}.$$

By this according to (4) and algebraic isomorphism between S and $\hat{S}$ we have:

$$s^{\nu-1}\frac{s^{(a-b)\nu}}{(s^\nu-\lambda)^a}e^{-\mu s^\nu}=\left\{\int_\mu^\infty \Phi(0,\ -\nu;\ -xt)\,K(a;\ b;\ \lambda(x-\mu))\frac{dx}{\nu x}\right\}.$$

Namely

$$\frac{s^{(a-b)\nu}}{(s^\nu-\lambda)^a}e^{-\mu s^\nu}=sl^\nu\left\{\int_\mu^t \Phi(0,\ -\nu;\ -xt^{-\nu})\,K(a;\ b;\ \lambda(x-\mu))\frac{dx}{\nu x}\right\}$$

$$=s\left\{\int_0^t\frac{(t-u)^{\nu-1}}{\Gamma(\nu)}du\int_\mu^\infty \Phi(0,\ -\nu;\ -xu^{-\nu})\,K(a;\ b;\ \lambda(x-\mu))\frac{dx}{\nu x}\right\}$$

$$=s\left\{\int_\mu^\infty K(a;\ b;\ \lambda(x-\mu))\,\Phi(1,\ -\nu;\ -xt^{-\nu})\,dx\right\}$$

$$=\left\{\int_\mu^\infty t^{-1}\Phi(0,\ -\nu;\ -xt^{-\nu})\,K(a;\ b;\ \lambda(x-\mu)\,dx\right\}.$$

Two integral equations in M

Proposition 1. Let $0<\nu<1$, $\lambda\in R$, $a,\ b\in C$, $\operatorname{Re} b>0$ $\operatorname{Re} a\geqslant 1/\nu$ and k be the least integer such that $k>\nu\ \operatorname{Re} b$. If $g=\{g(t)\}$ has a locally integrable derivate of order k and

$$g^{(m)}(0)=0\quad\text{for}\quad 0\leqslant m\leqslant k-1$$

then there exists one and only one locally integrable solution f of the equation (2). When $\nu\to 1$ then the solution of the equation (2) tends to the solution of the equation (1).

Proof. In terms of Mikusiński operators and according to Lemma 1 the integral equation (2) becomes the algebraic equation:

$$\frac{s^{(a-b)\nu}}{(s^\nu-\lambda)^a}f=g$$

which has the operator solution f equals

$$f=s^{(b-a)\nu}(s^\nu-\lambda)^a g$$

$$=s^{b\nu}g+\left[\left(I-\frac{\lambda}{s^\nu}\right)^a-I\right]s^{b\nu}g$$

$$=s^{b\nu}g+[(I-\lambda l^\nu)^a-I]\,s^{b\nu}g$$

where I is the unit operator in M. The latter form is useful for investigation of the conditions under which a locally integrable solution exists. First we note that $[(I-\lambda l^{\nu})^{a}-I]$ is an element from $\mathscr{L}$ because by the supposition $\operatorname{Re} a \geqslant 1/\nu$ so that $s^{b\nu} g$ must correspond to a locally integrable function. Using ideas from [2] and [5] if we write:

$$s^{b\nu} g = s^{-(k-b\nu)} s^{k} g = l^{k-b\nu} s^{k} g = l^{k-b\nu} \{g^{(k)}(t)\}$$

where k is given by supposition and according to the supposition about the function g it follows therefore that a locally integrable solution f of the integral equation (2) exists. The uniqueness of the solution follows from the uniqueness of the operator solution in the field M of operators as in [5]; of course, this means uniqueness among the appropriate equivalence classes which are the elements of the particular space of functions. A similar discussion holds for continuous solution.

Proposition 2. Under the suppositions of the Proposition 1. and Lemma 2. there exists one and only one solution f of the integral equation (3) in the field M. This solution f is not an element of $\mathscr{L}$ (or $\mathscr{C}$).

We omit the proof because it is in essence the same as for the Proposition 1. Namely, by Lemma 2 the integral equation (3) in the field M becomes the form

$$\frac{s^{(a-b)\nu}}{(s^{\nu}-\lambda)^{a}} e^{-\mu s^{\nu}} f = g$$

which has the operator solution f

$$f = e^{\mu s^{\nu}} s^{b\nu} g + e^{\mu s^{\nu}} [(I-l^{\nu}\lambda)^{a}-I] s^{b\nu} g.$$

By the supposition $\mu > 0$ it follows $e^{\mu s^{\nu}} \in M$ but $e^{\mu s^{\nu}} \notin \mathscr{L}$ (or $\mathscr{C}$) so that $f \in M$.

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