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LINEAR TRANSFORMATIONS OF TENSOR SPACES PRESERVING DECOMPOSABLE VECTORS

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Abstract. Let U_1, U_2, V_1, V_2 be vector spaces over a field F. Let $T \in \mathscr{L}(U_1 \otimes U_2, V_1 \otimes V_2)$ which maps decomposable vectors to decomposable vectors. Then it is shown that either (i) $T(U_1 \otimes U_2)$ consists entirely of decomposable vectors or (ii) there exist $\sigma \in S_2$, $f_1 \in \mathscr{L}(U_{\sigma(1)}, V_1)$ and $f_2 \in \mathscr{L}(U_{\sigma(2)}, V_2)$ such that $T(u_1 \otimes u_2) = f_1(u_{\sigma(1)}) \otimes f_2(u_{\sigma(2)})$ for all $u_1 \in U_1$, $u_2 \in U_2$. This result generalizes a theorem of Marcus and Moyls.

In [3], Marcus and Moyls determined the structure of linear transformations on the tensor product of two finite dimensional vector spaces preserving *non-zero* decomposable vectors. Westwick in [4] extended their result to the tensor product of m vector spaces. In this note we study linear transformations on the tensor product of two vector spaces preserving decomposable vectors.

Throughout this paper U_1, U_2, V_1, V_2 denote vector spaces over a field F. A subspace of the tensor space $U_1 \otimes U_2$ is called a *decomposable subspace* if all its elements are decomposable vectors. Two *non-zero* decomposable vectors $x_1 \otimes y_1$ and $x_2 \otimes y_2$ of $U_1 \otimes U_2$ are said to be *adjacent* if $\langle x_1 \rangle = \langle x_2 \rangle$ or $\langle y_1 \rangle = \langle y_2 \rangle$. It is known that the sum of two non-zero decomposable vectors is decomposable if and only if they are adjacent. Hence a subspace D of $U_1 \otimes U_2$ is decomposable if and only if

$$D = x \otimes N \equiv \{x \otimes y : y \in N\}$$

for some x in U_1 and some subspace N of U_2 or

$$D = M \otimes y \equiv \{x \otimes y : x \in M\}$$

for some y in U_2 and some subspace M of U_1.

In the following two lemmas, T denotes a linear map from $U_1 \otimes U_2$ to $V_1 \otimes V_2$ which takes decomposable vectors to decomposable vectors.

Lemma 1. *Let x_1 and x_2 be two vectors of U_1. Then there do not exist vectors $z \in V_1$, $w \in V_2$ and subspaces W of V_2, Z of V_1 such that*

$$T(x_1 \otimes U_2) = z \otimes W \neq 0$$

$$T(x_2 \otimes U_2) = Z \otimes w \neq 0$$

and $\langle z \rangle \neq Z$, $\langle w \rangle \neq W$.

Proof. Suppose the contrary. Let $z_1 \in Z$, $w_1 \in W$ such that $\langle z_1\rangle \neq \langle z\rangle$, $\langle w_1\rangle \neq \langle w\rangle$, $z_1 \neq 0$ and $w_1 \neq 0$. Then

$$T(x_1 \otimes y_1) = z \otimes w_1, \qquad T(x_2 \otimes y_2) = z_1 \otimes w$$

for some $y_1 \in U_1$, $y_2 \in U_2$. Let $T(x_1 \otimes y_2) = z \otimes w_2$, $T(x_2 \otimes y_1) = z_2 \otimes w$. Then since $T((x_1 + x_2) \otimes y_1)$ is decomposable it follows that $z_2 = 0$ or $\langle z_2\rangle = \langle z\rangle$. Similarly $T((x_1 + x_2) \otimes y_2)$ is decomposable implies $w_2 = 0$ or $\langle w_2\rangle = \langle w\rangle$. Hence $\dim\langle z, z_1 + z_2\rangle = 2$, $\dim\langle w, w_1 + w_2\rangle = 2$. Since

$$T((x_1 + x_2) \otimes (y_1 + y_2)) = z \otimes (w_1 + w_2) + (z_1 + z_2) \otimes w,$$

and $z \otimes (w_1 + w_2)$, $(z_1 + z_2) \otimes w$ are not adjecent, it follows that $T((x_1 + x_2) \otimes (y_1 + y_2))$ is not a decomposable vector. This contradicts the hypothesis. Hence the lemma is proved.

Lemma 2. *If* $T(x_1 \otimes U_2) = z_1 \otimes W_1 \neq 0$, $T(x_2 \otimes U_2) = z_2 \otimes W_2 \neq 0$ *where* z_1, z_2 *are linearly independent*, $\dim(W_1 + W_2) \geqslant 2$, *then* $W_1 = W_2$ *and there exists* $g \in \mathscr{L}(U_2, V_2)$ *such that for every* $y \in U_2$,

$$T(x_1 \otimes y) = z_1 \otimes g(y),$$

$$\langle T(x_2 \otimes y)\rangle = \langle z_2 \otimes g(y)\rangle.$$

Proof. Let $w_1 \in W_1$, $w_2 \in W_2$ such that w_1, w_2 are linearly independent. Choose $y_1, y_2 \in U_2$ such that

$$T(x_1 \otimes y_1) = z_1 \otimes w_1, \qquad T(x_2 \otimes y_2) = z_2 \otimes w_2.$$

Suppose $T(x_1 \otimes y_2) = z_1 \otimes w_1^*$, $T(x_2 \otimes y_1) = z_2 \otimes w_2^*$. Since $T((x_1 + x_2) \otimes y_1)$ is decomposable it follows that $w_2^* = 0$ or $\langle w_1\rangle = \langle w_2^*\rangle$. Also $T((x_1 + x_2) \otimes y_2)$ is decomposable implies $w_1^* = 0$ or $\langle w_1^*\rangle = \langle w_2\rangle$. By hypothesis, $T(x_1 + x_2) \otimes (y_1 + y_2)) = z_1 \otimes (w_1 + w_1^*) + z_2 \otimes (w_2 + w_2^*)$ is decomposable. Hence we have $\langle w_1\rangle = \langle w_2^*\rangle$, $\langle w_1^*\rangle = \langle w_2\rangle$. Thus $T((x_1 + x_2) \otimes y_1) = z_1 \otimes w_1 + \lambda_1 z_2 \otimes w_1$, $T((x_1 + x_2) \otimes y_2) = z_1 \otimes \lambda_2 w_2 + z_2 \otimes w_2$ for some non-zero λ_1, λ_2 in F. Since $T((x_1 + x_2) \otimes U_2)$ is a decomposable subspace, we then have

$$T((x_1 + x_2) \otimes U_2) = (z_1 + \lambda_1 z_2) \otimes W$$

for some subspace W of V_2 where $W \supseteq \langle w_1, w_2\rangle$.

Let $g \in \mathscr{L}(U_2, V_2)$ be defined as follows:

$$g(y) = \bar{y} \quad \text{if} \quad T(x_1 \otimes y) = z_1 \otimes \bar{y}.$$

Let $T(x_2 \otimes y) = z_2 \otimes \tilde{y}$. Since $T((x_1 + x_2) \otimes y) \in (z_1 + \lambda_1 z_2) \otimes W$, clearly we have $\langle \bar{y}\rangle = \langle \tilde{y}\rangle$. This implies that $W_1 = W_2$. Hence the lemma is proved.

We are now ready to prove our main result:

Theorem 1. *Let* $T: U_1 \otimes U_2 \to V_1 \otimes V_2$ *be a linear map which takes decomposable vectors to decomposable vectors. Then*

(i) $T(U_1 \otimes U_2)$ *is a decomposable subspace of* $V_1 \otimes V_2$

or

(ii) *there exist* $\sigma \in S_2$, $f_1 \in \mathcal{L}(U_{\sigma(1)}, V_1)$, $f_2 \in \mathcal{L}(U_{\sigma(2)}, V_2)$ *such that for all* $u_1 \in U_1$, $u_2 \in U_2$,

$$T(u_1 \otimes u_2) = f_1(u_{\sigma(1)}) \otimes f_2(u_{\sigma(2)}).$$

Proof. Assume that $T(U_1 \otimes U_2)$ is not a decomposable subspace of $V_1 \otimes V_2$. Then it is not hard to see that there exist vectors x_1, x_2 in U_1, y_1, y_2 in U_2, independent vectors z_1, z_2 in V_1 and independent vectors w_1, w_2 in V_2 such that

$$T(x_1 \otimes y_1) = z_1 \otimes w_1, \qquad T(x_2 \otimes y_2) = z_2 \otimes w_2.$$

Since $T(x_1 \otimes U_2)$ and $T(x_2 \otimes U_2)$ are decomposable subspaces, it follows from Lemma 1 that

(a) $T(x_1 \otimes U_2) = z_1 \otimes W_1$, $T(x_2 \otimes U_2) = z_2 \otimes W_2$, for some subspaces W_1, W_2 in V_2;

or

(b) $T(x_1 \otimes U_2) = H_1 \otimes w_1$, $T(x_2 \otimes U_2) = H_2 \otimes w_2$, for some subspaces H_1, H_2 in V_1.

Suppose (a) holds. In view of Lemma 2 there exists $g \in \mathcal{L}(U_2, V_2)$ such that for all $u_2 \in U_2$,

$$T(x_1 \otimes u_2) = z_1 \otimes g(u_2),$$

$$\langle T(x_2 \otimes u_2) \rangle = \langle z_2 \otimes g(u_2) \rangle$$

and $W_1 = W_2$. Let $x \in U_1$ such that $T(x \otimes U_2) \neq 0$. Since $\dim W_1 \geqslant 2$, it follows from Lemma 1 that $T(x \otimes U_2) = z_x \otimes W_x$ for some z_x in V_1 and some subspace W_x in V_2. Since $\dim(W_1 + W_x) \geqslant 2$, by Lemma 2, we have for every $u_2 \in U_2$,

(1) $$\langle T(x \otimes u_2) \rangle = \langle z_x \otimes g(u_2) \rangle.$$

Note that $T(x_1 \otimes y_1) = z_1 \otimes w_1$, $T(x_2 \otimes y_1) = \lambda z_2 \otimes w_1$ for some non-zero $\lambda \in F$. Hence $T(U_1 \otimes y_1) = M_1 \otimes w_1$ for some subspace M_1 of V_1. Similarly $T(U_1 \otimes y_2) = M_2 \otimes w_2$ for some subspace M_2 of V_1. By an analogue of Lemma 2, there exists $f \in \mathcal{L}(U_1, V_1)$ such that for all $u_1 \in U_1$,

$$T(u_1 \otimes y_1) = f(u_1) \otimes w_1, \qquad \langle T(u_1 \otimes y_2) \rangle = \langle f(u_1) \otimes w_2 \rangle$$

and $M_1 = M_2$. Let $y \in U_2$ such that $T(U_1 \otimes y) \neq 0$. By an analogue of Lemma 1, $T(U_1 \otimes y) = M_y \otimes w_y$ for some w_y in V_2 and some subspace M_y in V_1. Again by an analogue of Lemma 2, we have for all $u \in U_1$,

(2) $$\langle T(u_1 \otimes y) \rangle = \langle f(u_1) \otimes w_y \rangle.$$

Consequently from (1) and (2), we have for any $x \otimes y \notin \operatorname{Ker} T$,

$$T(x \otimes y) = \lambda_{x \otimes y} f(x) \otimes g(y)$$

for some $\lambda_{x \otimes y}$ in F.

Let $T(u_1 \otimes u_2) = 0$. We shall show that $f(u_1) = 0$ or $g(u_2) = 0$. Suppose that $g(u_2) \neq 0$ and $f(u_1) \neq 0$. Then

$$\begin{aligned} T(x_1 \otimes y_1) \neq 0 \Rightarrow \quad & T(x_1 \otimes U_2) = f(x_1) \otimes (\text{range } g) \neq 0 \\ \Rightarrow \quad & \langle T(x_1 \otimes u_2) \rangle = \langle f(x_1) \otimes g(u_2) \rangle \neq 0 \\ \Rightarrow \quad & T(U_1 \otimes u_2) = (\text{range } f) \otimes g(u_2) \neq 0 \\ \Rightarrow \quad & \langle T(u_1 \otimes u_2) \rangle = \langle f(u_1) \otimes g(u_2) \rangle \neq 0, \end{aligned}$$

which is a contradiction.

Hence $\langle T(x \otimes y) \rangle = \langle f(x) \otimes g(y) \rangle$ for all $x \in U_1$, $y \in U_2$. Let $\mathcal{D}$ denote the collection of all (A_1, A_2) where A_1, A_2 are adjacent decomposable vectors such that $T(A_1)$, $T(A_2)$ are linearly independent. It is easily shown that if $(A_1, A_2) \in \mathcal{D}$ then $\lambda_{A_1} = \lambda_{A_2}$. Now let $u_1 \otimes u_2$, $u_3 \otimes u_4$ be such that

$$T(u_1 \otimes u_2) \neq 0, \quad T(u_3 \otimes u_4) \neq 0 \quad \text{and} \quad (u_1 \otimes u_2,\ u_3 \otimes u_4) \notin \mathcal{D}.$$

Since $\dim(\text{range } f) \geqslant 2$, $\dim(\text{range } g) \geqslant 2$, it is not hard to construct decomposable vectors $B_1, \ldots, B_j$ $(j \leqslant 3)$ such that

$$(u_1 \otimes u_2,\ B_1) \in \mathcal{D}, \quad (B_1, B_2) \in \mathcal{D}, \ldots, \quad (B_j,\ u_3 \otimes u_4) \in \mathcal{D}.$$

Hence $\lambda_{u_1 \otimes u_2}, = \lambda_{B_1} = \cdots = \lambda_{u_3 \otimes u_4}$. Consequently there exists $c \in F$ such that $T = cf \otimes g$.

Suppose (b) holds. Let $S: V_1 \otimes V_2 \to V_2 \otimes V_1$ be the canonical isomorphism with $S(v_1 \otimes v_2) = v_2 \otimes v_1$. Then $ST: U_1 \otimes U_2 \to V_2 \otimes V_1$ maps decomposable vectors to decomposable vectors and $ST(x_1 \otimes U_2) = w_1 \otimes H_1$, $ST(x_2 \otimes U_2) = w_2 \otimes H_2$. In view of case (a), there exist $f \in \mathcal{L}(U_1, V_2)$, $g \in \mathcal{L}(U_2, V_1)$ such that $ST = f \otimes g$. Hence $T(u_1 \otimes u_2) = g(u_2) \otimes f(u_1)$ for all $u_1 \in U$, $u_2 \in U_2$.

This completes the proof of the theorem.

In matrix language we have the following.

Corollary. *Let $M_{m,n}(F)$ be the space of all $m \times n$ matrices over F. Let $T: M_{m,n}(F) \to M_{m,n}(F)$ be a linear map such that rank $A \leqslant 1$ implies rank $T(A) \leqslant 1$. Then*

(i) *$T(M_{m,n}(F)) - \{0\}$ consists entirely of rank one matrices.*

or

(ii) *there exist $P \in M_{m,m}(F)$, $Q \in M_{n,n}(F)$ such that $T(A) = PAQ$ for all $A \in M_{m,n}(F)$;*

or

(iii) *there exist $C, D \in M_{m,n}(F)$ such that $T(A) = CA'D$ for all $A \in M_{m,n}(F)$.*

The following result of Marcus and Moyls [3] can be derived from Theorem 1.

Theorem 2. *Let U_1, U_2 be finite dimensional vector spaces over an algebraically closed field F. Let $T \in \mathcal{L}(U_1 \otimes U_2,\ U_1 \otimes U_2)$. If T maps non-zero decomposable vectors to non-zero decomposable vectors, then there exist $\sigma \in S_2$ and non-singular maps*

$$f_1 \in \mathcal{L}(U_{\sigma(1)}, U_1), \qquad f_2 \in \mathcal{L}(U_{\sigma(2)}, U_2)$$

such that

$$T(u_1 \otimes u_2) = f_1(u_{\sigma(1)}) \otimes f_2(u_{\sigma(2)}) \quad \textit{for all} \quad u_1 \in U_1, \quad u_2 \in U_2.$$

Proof. The result is trivial when $\dim U_1 = 1$ or $\dim U_2 = 1$. Let $\dim U_1 \geqslant 2$, $\dim U_2 \geqslant 2$. Assume that $T(U_1 \otimes U_2)$ is a decomposable subspace. Then

(a) $T(U_1 \otimes U_2) = z \otimes W_2$ for some $z \in U_1$, $W_2 \subseteq U_2$ or

(b) $T(U_1 \otimes U_2) = W_1 \otimes y$ for some $y \in U_2$, $W_1 \subseteq U_1$.

If (a) holds, then for any non-zero vector u in U_1, $T(u \otimes U_2) \subseteq z \otimes W_2$. This implies that $T(u \otimes U_2) = z \otimes U_2$ otherwise T maps some non-zero decomposable vectors to the zero vector. Let x_1 and x_2 be two linearly independent vectors in U_1. Then $T(x_1 \otimes U_2) = T(x_2 \otimes U_2) = z \otimes U_2$. Let $f_i : U_2 \to U_2$ be non-singular linear maps such that $T(x_i \otimes v) = z \otimes f_i(v)$, $v \in U_2$, $i = 1,2$. Since F is algebraically closed, $f_2^{-1} f_1(w) = \lambda w$ for some non-zero eigenvector $w \in U_2$ where λ is an eigenvalue of $f_2^{-1} f_1$. Consequently

$$T\big((x_1 - \lambda x_2) \otimes w\big) = z \otimes f_1(w) - \lambda z \otimes f_2(w)$$
$$= z \otimes \big(f_1(w) - \lambda f_2(w)\big) = 0.$$

This contradicts the hypothesis on T since $(x_1 - \lambda x_2) \otimes w \neq 0$. Similarly case (b) leads to a contradiction. Hence by Theorem 1 there exist $\sigma \in S_2$ and non-singular maps $f_1 \in \mathscr{L}(U_{\sigma(1)}, U_1)$, $f_2 \in \mathscr{L}(U_{\sigma(2)}, U_2)$ such that $T(u_1 \otimes u_2) = f_1(u_{\sigma(1)}) \otimes f_2(u_{\sigma(2)})$ for all $u_1 \in U_1$, $u_2 \in U_2$.

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