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ON SOME CLASSES OF LINEAR FUNCTION TRANSFORMATIONS OF SEQUENCES (II)

(Inclusion theorems)

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Preliminaries. This is a sequel of the research started in [1] (see also [2] and [3]). Namely, let X be located in a topological space and let x' be an accumulation point of X possessing a countable neighbourhood base. Suppose $F=(f_j)$, $f_j=f_j(x)$ $(j=0, 1, 2, \dots)$, is a sequence of functions from X to the real (complex) scalars. Without restriction we can assume $x' \notin X$. Given $t=(t_j) \in T$[1], let $t=(t_j)\in (X, F)^d_{gx'}$ if there exists a neighbourhood 0 of x' such that the series $(x, F)(t)=\sum_{j=0}^{\infty} f_j(x)\, t_j$ converges for every $x \in X \cap 0$.

If $t=(t_j)\in (X, F)^d_{gx'}$, denote the set $\left\{x\in X \,\middle|\, \sum_{j=0}^{\infty} f_j(x)\, t_j \text{ converges}\right\}$ by X_t and put

$$(X, F)_{gx'}(t)=\begin{pmatrix} x \\ (x, F)(t) \end{pmatrix}_{x\in X_t}.$$

The function thus obtained $(X, F)_{gx'}(t)$ is called the *transform* of the sequence $t=(t_j)$ by the *operator* $(X, F)_{gx'}$; then the set $(X, F)^d_{gx'}$ is called the *applicability domain* of the operator $(X, F)_{gx'}$. The set of all $t=(t_j)$'s such that there exists a neighbourhood 0 of x' with the property

$$\sup_{x\in X\cap 0} \left| \sum_{j=0}^{\infty} f_j(x)\, t_j \right| < \infty,$$

is called the *b-applicability domain* of the operator $(X, F)_{gx'}$ and is denoted by $(X, F)^b_{gx'}$.

Obviously, the operator $(X, F)_{gx'}$ is an extension of the corresponding operator (X, F) (see [3] or [1]). Taking into account the definition of the applicability domain and the b-applicability domain of the operator (X, F) we can set

$$(X, F)^d_{gx'} = \{t=(t_j) \mid (\exists \text{ a neighbourhood } 0 \text{ of } x')\, (t=(t_j)\in (X\cap 0, F)^d)\}$$

[1] T denotes the set of all sequences.

and

$$(X,\ F)^{b}_{gx'}=\{t=(t_j)\,|\,(\exists \text{ a neighbourhood } 0 \text{ of } x')\,(t=(t_j)\in(X\cap 0,\ F)^{b})\}.$$

When for $t=(t_j)\ (\in(X,\ F)^{d}_{gx'})$ there exists

$$\lim_{X_t\ni x\to x'} (x,\ F)\,(t),$$

we say that this number is the *limit* of the sequence $t=(t_j)$ by the *method* $(X,\ F,\ x')_g$ and denote it by $(X,\ F,\ x')_g(t)$ or by $(X,\ F,\ x')_g-\lim t_j$. The set of all $t=(t_j)$'s for which $(X,\ F,\ x')_g-\lim t_j$ exists is called the *applicability domain*, or the *convergence domain*, of the method $(X,\ F,\ x')_g$ and is denoted by $(X,\ F,\ x')_g{}^c$; the set of all $t=(t_j)$'s from $(X,\ F,\ x')_g{}^c$ with

$$(X,\ F,\ x')_g-\lim t_j=0$$

is called the *o-applicability domain*, or the *o-convergence domain*, of the method $(X,\ F,\ x')_g$ and is denoted by $(X,\ F,\ x')_g{}^o$.

The method $(X,\ F,\ x')_g$ is evidently a generalization of the corresponding method $(X,\ F,\ x')$ (see [3] or [1]). In the notations $(X,\ F,\ x')^c$ and $(X,\ F,\ x')^o$, we can, equivalently, take:

$$(X,\ F,\ x')_g{}^c=\{t=(t_j)\,|\,(\exists \text{ a neighbourhood } 0 \text{ of } x')\,(t=(t_j)\in(X\cap 0,\ F,\ x')^c)\}$$

and

$$(X,\ F,\ x')_g{}^o=\{t=(t_j)\,|\,(\exists \text{ a neighbourhood } 0 \text{ of } x')\,(t=(t_j)\in(X\cap 0,\ F,\ x')^o)\}.$$

In [2], Section 4, we quoted results concerning the necessary and sufficient conditions under which $(X,\ F)^{d}_{gx'}$ and $(X,\ F)^{b}_{gx'}$ include T^o, T^c, T^b [1] and T. In [2], Section 5, we find such results for the sets $(X,\ F,\ x')_g{}^c$ and $(X,\ F,\ x')_g{}^o$.

S u m m a r y. Some special classes of linear transformations of the type above exposed are delt with in the present paper. Namely, in Section 3 we introduce the notion of an operator $(X,\ F)_{gx'}$ of the *type C* and provide necessary and sufficient conditions in order that $(X,\ F)^{d}_{gx'}$ and $(X,\ F)^{b}_{gx'}$ of such an operator $(X,\ F)_{gx'}$ include a GK-space[2]. Matrix operators in the new generalized sense are of the *type C*. Therefore, by specializations of the GK-space we may obtain results (either known or not) for matrix operators of such a type. In Section 4 we propose the definition of the *continuous* method $(X,\ F,\ x')_g$, having matrix methods of the generalized type as a special case. Then we disscuss conditions necessary and sufficient that $(X,\ F,\ x')_g{}^c$ and $(X,\ F,\ x')_g{}^o$ of such a method $(X,\ F,\ x')_g$ include a GK-space. From these results, by specializations, we can obtain either quite new theorems or generalizations of the known ones for matrix methods. We also observe that both above mentioned definitions are founded on the notion of the continuous method $(X,\ F,\ x')$ (consequently, on the notion of the operator $(X,\ F)$ of the type C) (see [1]).

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1) i.e. the sets of null-convergent, convergent and bounded sequences, respectively

2) that is: a complete metric group in which convergence by metric implies the one by all the coordinates

3. Operators $(X, F)_g$ of the type C

Definition 3.1. An operator $(X, F)_{gx'}$ is said to be of the *type C* if for every neighbourhood 0 of x' there exists a neighbourhood $0_* \subseteq 0$ of the same point such that the method $(X \cap 0_*, F, x')$ is continuous (see [1], Definitions 2.1 and 1.1). (As in [1], Definition 2.1, we assume that X is a part of some Hausdorff topological space.)

Remark 3.1. Matrix operators $(F)_g$, $F=(f_{ij})$, are operators $(X, F)_{gx'}$ of the *type C*. Namely, then we have $X=\{0, 1, \dots, i, \dots\}$, $x'=+\infty$ and $F=(f_j)$ with $f_j(i)=f_{ij}$ $(i, j=0, 1, 2, \dots)$, and it is easy to see that in the considered case for every neighbourhood 0 of $x'=\infty$ the method $(X \cap 0, F, x')$ is a matrix, and consequently a continuous method (see [1], Remark 2.4 and Definition 2.1).

Remark 3.2. Operators $([x_0, x'), F)_{gx'}$ such that the corresponding methods $([x_0, x'), F, x')$ are W-continuous (see [4], Definition 8, and [1], Remark 2.5) represent new examples of continuous operators $(X, F)_g$. Indeed, when a neighbourhood 0 of x' is given, as a neighbourhood $0_* \subseteq 0$ of x' we can take such a neighbourhood that $[x_0, x') \cap 0_* = [x_{i_0}, x')$, where $x_{i_0} \in 0$ is a member of the sequence (x_i) from Włodarski's definition of the continuous method $([x_0, x'), F, x')$. The definition of the W-continuous method $([x_0, x'), F, x')$ implies that the method $([x_{i_0}, x'), F, x')$ thus obtained is continuous.

Theorem 3.1. *Let T_1 be a connected GK-space and let $(X, F)_{gx'}$ be an operator of the type C. Then the following equivalence is valid:*

$T_1 \subseteq (X, F)^d_{gx'}$ if and only if there exists a neighbourhood 0_* of x' such that the method $(X \cap 0_*, F, x')$ is continuous (see [1], Definitions 1.1. and 2.1) *and satisfies the inclusion $T_1 \subseteq (X \cap 0_*, F)^d$.*

Remark 3.3. Specializing T_1 by T^0, T^c, T^b and T, and taking into account the analogous results for operators (X, F) of the type C[1)], we can obtain necessary and sufficient conditions under which $(X, F)^d_{gx'}$ of a type C operator $(X, F)_{gx'}$ includes T^o, T^c, T^b and T, respectively. Analogous remarks are valid for other theorems of this paper, too.

Proof of Theorem 3.1. Assume $T_1 \subseteq (X, F)^d_{gx'}$. According to [2], Lemma 4.3, there exists a neighbourhood 0 of x' with the property $T_1 \subseteq (X \cap 0, F)^d$. Since the operator $(X, F)_{gx'}$ is of the type C, there is a neighbourhood $0_* \subseteq 0$ of x' such that the method $(X \cap 0_*, F, x')$ is continuous. Now, the inclusion $T_1 \subseteq (X \cap 0_*, F)^d$ follows from $(X \cap 0, F)^d \subseteq (X \cap 0_*, F)^d$. The second implication is evident (this part of our theorem is valid without the assumption that $(X, F)_{gx'}$ is of the type C). The proof of Theorem 3.1 is complete.

Theorem 3.2. *Let T_1 be a connected GK-space and let an operator $(X, F)_{gx'}$ be of the type C. Then $T_1 \subseteq (X, F)^b_{gx'}$ if and only if there exists a neighbourhood 0_* of x' such that the method $(X \cap 0_*, F, x')$ is continuous and the inclusion $T_1 \subseteq (X \cap 0_*, F)^b$ is valid.*

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1) If a method (X, F, x') is continuous, the corresponding operator (X, F) is of the type C.

Proof. Similarly to the proof of Theorem 3.1, the inclusion $T_1 \subseteq (X, F)^b_{gx'}$ (by $(X, F)^b_{gx'} \subseteq (X, F)^d_{gx'}$) implies the existence of a neighbourhood 0_* of x' such that the method $(X \cap 0_*, F, x')$ is continuous and $T_1 \subseteq (X \cap 0_*, F)^d$. We shall show the validity of $T_1 \subseteq (X \cap 0_*, F)^b$. Indeed, assume $t = (t_j) \in T_1$. Then $T_1 \subseteq (X, F)^b_{gx'}$ implies the existence of a neighbourhood $0_1 \subseteq 0_*$ of x' with the property $\sup_{x \in X \cap 0_1} \left| \sum_{j=0}^{\infty} f_j(x) t_j \right| < \infty$. At the same time the continuity of the method $(X \cap 0_*, F, x')$ and $t = (t_j) \in (X \cap 0_*, F)^d$ imply

$$\sup_{x \in X \cap 0_* - 0_1} \left| \sum_{j=0}^{\infty} f_j(x) t_j \right| < \infty$$

(see the footnote on the preceding side and [1], Remark 1.3). Consequently, $\sup_{x \in X \cap 0_*} \left| \sum_{j=0}^{\infty} f_j(x) t_j \right| < \infty$, i.e $t = (t_j) \in (X \cap 0_*, F)^b$. Since the second implication of our theorem is obvious, its proof is complete.

4. Continuous methods $(X, F, x')_g$

Definition 4.1. Let X be located in some Hausdorff topological space. Then the method $(X, F, x')_g$ is said to be *continuous* if the corresponding operator $(X, F)_{gx'}$ is of the type C (see Definition 3.1).

Remark 4.1. According to Remarks 3.1. and 3.2, the matrix methods and methods $([x_0, x'), F, x')_g$, such that the corresponding methods $([x_0, x'), F, x')$ are W-continuous, make subclasses of the class of continuous methods $(X, F, x')_g$.

Observe that a method $(X, F, x')_g$ can be continuous even in the case when the corresponding method (X, F, x') is not. The method (X, F, x') from [1], Remark 1.5, is of such a kind. Namely, in the commentary of Theorem 2.1 [1] we showed that this method is not continuous. At the same time, for every neighbourhood 0 of $x' = \infty$ which does not contain points of $[-3, -1]$ the method $(X \cap 0, F, x')$ is a matrix, and consequently a continuous method. Hence the operator $(X, F)_{gx'}$ is of the type C, and therefore the method $(X, F, x')_g$ is continuous.

Theorem 4.1. *Suppose T_1 is a connected GK-space and $(X, F, x')_g$ is continuous. Then $T_1 \subseteq (X, F, x')_g^c$ ($(X, F, x')_g^o$) is valid if and only if there exists a neighborhood 0_* of x' such that the method $(X \cap 0_*, F, x')$ is continuous and $T_1 \subseteq (X \cap 0_*, F, x')^c$ ($(X \cap 0_*, F, x')^o$) is true.*

Proof. We shall prove only the nontrivial implication. Accordingly, assume $T_1 \subseteq (X, F, x')_g^c$. By $(X, F, x')_g^c \subseteq (X, F)^d_{gx'}$ and [2], Lemma 4.3, there exists a neighbourhood 0 of x' such that $T_1 \subseteq (X \cap 0, F)^d$, whence $T_1 \subseteq (X \cap 0, F, x')^c$. Continuity of the method $(X, F, x')_g$ implies the existence of a neighbourhood $0_* \subseteq 0$ such that the method $(X \cap 0_*, F, x')$ is continuous (see Definitions 3.1 and 4.1). Now the inclusion $T_1 \subseteq (X \cap 0_*, F, x')^c$ follows from $(X \cap 0, F, x')^c \subseteq (X \cap 0_*, F, x')^c$. The implication in the case of the set $(X, F, x')_g^o$ is obtainable in a similar way (observe only that $T_1 \subseteq (X, F, x')_g^o$ and $T_1 \subseteq (X \cap 0, F, x')^c$ imply $T_1 \subseteq (X \cap 0, F, x')^o$).

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