ON SOME CLASSES OF LINEAR FUNCTION TRANSFORMATIONS OF SEQUENCES (II)

(Inclusion theorems)

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Preliminaries. This is a sequel of the research started in [1] (see also [2] and [3]). Namely, let $X$ be located in a topological space and let $x'$ be an accumulation point of $X$ possessing a countable neighbourhood base. Suppose $F = (f_j), f_j = f_j(x)$ $(j = 0, 1, 2, \ldots)$, is a sequence of functions from $X$ to the real (complex) scalars. Without restriction we can assume $x' \not\in X$. Given $t = (t_j) \in T$, let $t = (t_j) \in (X, F)^d_{x'}$ if there exists a neighbourhood $0$ of $x'$ such that the series $(x, F)(t) = \sum_{j=0}^{\infty} f_j(x) t_j$ converges for every $x \in X \cap 0$.

If $t = (t_j) \in (X, F)^d_{x'}$, denote the set $\left\{ x \in X \bigg| \sum_{j=0}^{\infty} f_j(x) t_j \text{ converges} \right\}$ by $X_t$ and put

$$(X, F)_{x'}(t) = \left( \frac{x}{(x, F)(t)} \right)_{x \in X_t}.$$  

The function thus obtained $(X, F)_{x'}(t)$ is called the transform of the sequence $t = (t_j)$ by the operator $(X, F)_{x'}$; then the set $(X, F)^d_{x'}$ is called the applicability domain of the operator $(X, F)_{x'}$. The set of all $t = (t_j)$'s such that there exists a neighbourhood $0$ of $x'$ with the property

$$\sup_{x \in X \cap 0} \left| \sum_{j=0}^{\infty} f_j(x) t_j \right| < \infty,$$

is called the b-applicability domain of the operator $(X, F)_{x'}$ and is denoted by $(X, F)_{b_{x'}}$.

Obviously, the operator $(X, F)_{x'}$ is an extension of the corresponding operator $(X, F)$ (see [3] or [1]). Taking into account the definition of the applicability domain and the b-applicability domain of the operator $(X, F)$ we can set

$$(X, F)^d_{x'} = \{ t = (t_j) \mid (\exists \text{ a neighbourhood } 0 \text{ of } x') (t = (t_j) \in (X \cap 0, F)^d) \}$$

1) $T$ denotes the set of all sequences.
and
\[(X, F)_{g^{x'}}^b = \{ t = (t_j) \mid (\exists \text{ a neighbourhood } 0 \text{ of } x') (t = (t_j) \subseteq (X \cap 0, F)_b) \} \]

When for \( t = (t_j) \subseteq (X, F)_{g^x} \) there exists
\[\lim_{x \uparrow x \to x'} (x, F)(t),\]
we say that this number is the limit of the sequence \( t = (t_j) \) by the method \((X, F, x')_g\) and denote it by \((X, F, x')_g(t)\) or by \((X, F, x')_g \lim t_j\). The set of all \( t = (t_j) \)'s for which \((X, F, x')_g \lim t_j\) exists is called the applicability domain, or the convergence domain, of the method \((X, F, x')_g\) and is denoted by \((X, F, x')_g^c\); the set of all \( t = (t_j) \)'s from \((X, F, x')_g^c\) with
\[(X, F, x')_g \lim t_j = 0\]
is called the o-applicability domain, or the o-convergence domain, of the method \((X, F, x')_g\) and is denoted by \((X, F, x')_g^o\).

The method \((X, F, x')_g\) is evidently a generalization of the corresponding method \((X, F, x')\) (see [3] or [1]). In the notations \((X, F, x')^c\) and \((X, F, x')^o\), we can, equivalently, take:
\[(X, F, x')_g^c = \{ t = (t_j) \mid (\exists \text{ a neighbourhood } 0 \text{ of } x') (t = (t_j) \subseteq (X \cap 0, F, x')^c) \}\]
and
\[(X, F, x')_g^o = \{ t = (t_j) \mid (\exists \text{ a neighbourhood } 0 \text{ of } x') (t = (t_j) \subseteq (X \cap 0, F, x')^o) \}\].

In [2], Section 4, we quoted results concerning the necessary and sufficient conditions under which \((X, F)_{g^x}^d\) and \((X, F)_{g^x}^b\) include \(T^o, T^c, T^b\)[1] and \(T\). In [2], Section 5, we find such results for the sets \((X, F, x')_g^c\) and \((X, F, x')_g^o\).

Summary. Some special classes of linear transformations of the type above exposed are dealt with in the present paper. Namely, in Section 3 we introduce the notion of an operator \((X, F)_{g^x}^c\) of the type C and provide necessary and sufficient conditions in order that \((X, F)_{g^x}^d\) and \((X, F)_{g^x}^b\) of such an operator \((X, F)_{g^x}\) include a GK-space\(^2\). Matrix operators in the new generalized sense are of the type C. Therefore, by specializations of the GK-space we may obtain results (either known or not) for matrix operators of such a type. In Section 4 we propose the definition of the continuous method \((X, F, x')_g\), having matrix methods of the general type as a special case. Then we discuss conditions necessary and sufficient that \((X, F, x')_g^c\) and \((X, F, x')_g^o\) of such a method \((X, F, x')_g\) include a GK-space. From these results, by specializations, we can obtain either quite new theorems or generalizations of the known ones for matrix methods. We also observe that both above-mentioned definitions are founded on the notion of the continuous method \((X, F, x')\) (consequently, on the notion of the operator \((X, F)\) of the type C) (see [1]).

\(^1\) i.e. the sets of null-convergent, convergent and bounded sequences, respectively

\(^2\) that is: a complete metric group in which convergence by metric implies the one by all the coordinates
3. Operators \((X, F)_{g'}\) of the type \(C\)

**Definition 3.1.** An operator \((X, F)_{g'}\) is said to be of the type \(C\) if for every neighbourhood 0 of \(x'\) there exists a neighbourhood 0* \(\subseteq 0\) of the same point such that the method \((X \cap 0_*, F, x')\) is continuous (see [1], Definitions 2.1 and 1.1). (As in [1], Definition 2.1, we assume that \(X\) is a part of some Hausdorff topological space.)

**Remark 3.1.** Matrix operators \((F)_{g'}\), \(F = (f_{ij})\), are operators \((X, F)_{g'}\) of the type \(C\). Namely, then we have \(X = \{0, 1, \ldots, i, \ldots\}\), \(x' = + \infty\) and \(F = (f_{ij})\) with \(f_{ij}(i) = f_{ij}(i, j = 0, 1, 2, \ldots)\), and it is easy to see that in the considered case for every neighbourhood 0 of \(x' = \infty\) the method \((X \cap 0, F, x')\) is a matrix, and consequently a continuous method (see [1], Remark 2.4 and Definition 2.1).

**Remark 3.2.** Operators \(((x_0, x'), F)_{g'}\) such that the corresponding methods \(((x_0, x'), F, x')\) are \(W\)-continuous (see [4], Definition 8, and [1], Remark 2.5) represent new examples of continuous operators \((X, F)_{g'}\). Indeed, when a neighbourhood 0 of \(x'\) is given, as a neighbourhood 0* \(\subseteq 0\) of \(x'\) we can take such a neighbourhood that \([x_0, x') \cap 0_0 = [x_0, x')\), where \(x_0 \in 0\) is a member of the sequence \((x_i)\) from Wlodarski's definition of the continuous method \(((x_0, x'), F, x')\). The definition of the \(W\)-continuous method \(((x_0, x'), F, x')\) implies that the method \(((x_0, x'), F, x')\) thus obtained is continuous.

**Theorem 3.1.** Let \(T_1\) be a connected GK-space and let \((X, F)_{g'}\) be an operator of the type \(C\). Then the following equivalence is valid:

\[T_1 \subseteq (X, F)_{g'}\] if and only if there exists a neighbourhood 0* of \(x'\) such that the method \((X \cap 0_*, F, x')\) is continuous (see [1], Definitions 1.1 and 2.1) and satisfies the inclusion \(T_1 \subseteq (X \cap 0_*, F)'\).

**Remark 3.3.** Specializing \(T_1\) by \(T^0, T^c, T^b\) and \(T\), and taking into account the analogous results for operators \((X, F)\) of the type \(C\)' (1), we can obtain necessary and sufficient conditions under which \((X, F)_{g'}\) of a type \(C\) operator \((X, F)_{g'}\) includes \(T^0, T^c, T^b\) and \(T\), respectively. Analogous remarks are valid for other theorems of this paper, too.

**Proof of Theorem 3.1.** Assume \(T_1 \subseteq (X, F)_{g'}\). According to [2], Lemma 4.3, there exists a neighbourhood 0 of \(x'\) with the property \(T_1 \subseteq (X \cap 0, F)'\). Since the operator \((X, F)_{g'}\) is of the type \(C\), there is a neighbourhood 0* \(\subseteq 0\) of \(x'\) such that the method \((X \cap 0_*, F, x')\) is continuous. Now, the inclusion \(T_1 \subseteq (X \cap 0_*, F)'\) follows from \((X \cap 0, F)' \subseteq (X \cap 0_*, F)'\). The second implication is evident (this part of our theorem is valid without the assumption that \((X, F)_{g'}\) is of the type \(C\)). The proof of Theorem 3.1 is complete.

**Theorem 3.2.** Let \(T_1\) be a connected GK-space and let an operator \((X, F)_{g'}\) be of the type \(C\). Then \(T_1 \subseteq (X, F)_{g'}\) if and only if there exists a neighbourhood 0* of \(x'\) such that the method \((X \cap 0_*, F, x')\) is continuous and the inclusion \(T_1 \subseteq (X \cap 0_*, F)'\) is valid.

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1) If a method \((X, F, x')\) is continuous, the corresponding operator \((X, F)\) is of the type \(C\).
Proof. Similarly to the proof of Theorem 3.1, the inclusion $T_1 \subseteq (X, F)_{x^e}^b$ (by $(X, F)_{x^e}^b \subseteq (X, F)_{x^e}^d$) implies the existence of a neighbourhood $0_\ast^x$ of $x'$ such that the method $(X \cap 0_\ast^x, F, x')$ is continuous and $T_1 \subseteq (X \cap 0_\ast^x, F)^d$. We shall show the validity of $T_1 \subseteq (X \cap 0_\ast^x, F)^b$. Indeed, assume $t = (t_j) \subseteq T_1$. Then $T_1 \subseteq (X, F)_{x^e}^b$ implies the existence of a neighbourhood $0_1 \subseteq 0_\ast^x$ of $x'$ with the property \[
abla \sup_{x \in X \cap 0_1} \left| \sum_{j=0}^{\infty} f_j(x) t_j \right| < \infty \quad \text{At the same time the continuity of the method} \quad (X \cap 0_\ast^x, F, x') \quad \text{and} \quad t = (t_j) \subseteq (X \cap 0_\ast^x, F)^d \quad \text{imply} \quad \sup_{x \in X \cap 0_1} \left| \sum_{j=0}^{\infty} f_j(x) t_j \right| < \infty \quad \text{(see the footnote on the preceding side and \cite{1}, Remark 1.3). Consequently,} \quad \sup_{x \in X \cap 0_1} \left| \sum_{j=0}^{\infty} f_j(x) t_j \right| < \infty, \quad \text{i.e.} \quad t = (t_j) \subseteq (X \cap 0_\ast^x, F)^b. \quad \text{Since the second implication of our theorem is obvious, its proof is complete.}

4. Continuous methods $(X, F, x')_g$

Definition 4.1. Let $X$ be located in some Hausdorff topological space. Then the method $(X, F, x')_g$ is said to be continuous if the corresponding operator $(X, F)_{x^e}^g$ is of the type $C$ (see Definition 3.1).

Remark 4.1. According to Remarks 3.1. and 3.2, the matrix methods and methods $([x_0, x'), (F, x')_g)$, such that the corresponding methods $([x_0, x'), F, x')$ are $W$-continuous, make subclasses of the class of continuous methods $(X, F, x')_g$.

Observe that a method $(X, F, x')_g$ can be continuous even in the case when the corresponding method $(X, F, x')$ is not. The method $(X, F, x')$ from \cite{1}, Remark 1.5, is of such a kind. Namely, in the commentary of Theorem 2.1 \cite{1} we showed that this method is not continuous. At the same time, for every neighbourhood $0$ of $x' = \infty$ which does not contain points of $[-3, -1]$ the method $(X \cap 0, F, x')$ is a matrix, and consequently a continuous method. Hence the operator $(X, F)_{x^e}^g$ is of the type $C$, and therefore the method $(X, F, x')_g$ is continuous.

Theorem 4.1. Suppose $T_1$ is a connected GK-space and $(X, F, x')_g$ is continuous. Then $T_1 \subseteq (X, F, x')_g^c$ (or $(X, F, x')_g^{e^c}$) is valid if and only if there exists a neighbourhood $0_\ast^x$ of $x'$ such that the method $(X \cap 0_\ast^x, F, x')$ is continuous and $T_1 \subseteq (X \cap 0_\ast^x, F, x')^c$ (or $(X \cap 0_\ast^x, F, x')^{e^c}$) is true.

Proof. We shall prove only the nontrivial implication. Accordingly, assume $T_1 \subseteq (X, F, x')_g^c$. By $(X, F, x')_g^c \subseteq (X, F)_{x^e}^d$ and \cite{2}, Lemma 4.3, there exists a neighbourhood $0$ of $x'$ such that $T_1 \subseteq (X \cap 0, F)^d$, whence $T_1 \subseteq (X \cap 0, F, x')^c$. Continuity of the method $(X, F, x')_g$ implies the existence of a neighbourhood $0_\ast^x \subseteq 0$ such that the method $(X \cap 0_\ast^x, F, x')$ is continuous (see Definitions 3.1 and 4.1). Now the inclusion $T_1 \subseteq (X \cap 0_\ast^x, F, x')^c$ follows from $(X \cap 0, F, x')^c \subseteq (X \cap 0_\ast^x, F, x')^c$. The implication in the case of the set $(X, F, x')_g^{e^c}$ is obtainable in a similar way (observe only that $T_1 \subseteq (X, F, x')_g^{e^c}$ and $T_1 \subseteq (X \cap 0, F, x')^c$ imply $T_1 \subseteq (X \cap 0, F, x')^{e^c}$).
REFERENCES


