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ON SOME CLASSES OF LINEAR FUNCTION TRANSFORMATIONS OF SEQUENCES (I)

(Inclusion theorems)

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Preliminaries. Let X be an amorphous set and let $F=(f_j)$, $f_j=f_j(x)$ $(j=0, 1, 2, \ldots)$, be a sequence of functions from X to the real (complex) scalars. With $t=(t_j)\in T$, where T denotes the set of all real (complex) sequences, and $x\in X$, we set (formally)

$$(x, F)(t)=\sum_{j=0}^{\infty} f_j(x)\, t_j$$

and

$$(X, F)(t)=\begin{pmatrix} x \\ (x, F)(t) \end{pmatrix}_{x\in X}.$$

In [1] we have introduced the notion of the operator (X, F) as follows.

If for a fixed $t=(t_j)$ the series $(x, F)(t)$ converges for every $x\in X$, then the function obtained $(X, F)(t)$ is called a *transform* of the sequence $t=(t_j)$ by the *operator* (X, F). The set of all sequences $t=(t_j)$ such that the transform $(X, F)(t)$ exists, is called the *applicability domain* of the operator (X, F) and is denoted by $(X, F)^d$; the set of all sequences $t=(t_j)$ from $(X, F)^d$ for which the transform $(X, F)(t)$ is bounded (on X), is said to be the *b-applicability domain* of the operator (X, F) and is denoted by $(X, F)^b$.

Let X be located in a topological space and possess an accumulation point x' with a countable neighbourhood base. It is not a restriction, if we assume $x'\notin X$. In [1] we have proposed the below cited definition of the convergence method (X, F, x').

If for $t=(t_j)\in(X, F)^d$ there exists

$$\lim_{X\ni x\to x'} (x, F)(t),$$

it is called the *limit* of the sequence $t=(t_j)$ by the *method* (X, F, x') and denoted by $(X, F, x')(t)$ or by $(X, F, x')-\lim t_j$. The set of all $t=(t_j)$'s for which $(X, F, x')-\lim t_j$ exists is called the *applicability domain*, or the *convergence domain*, of the method (X, F, x') and is denoted by $(X, F, x')^c$; the set of all $t=(t_j)$'s such that $(X, F, x')-\lim t_j=0$ is said to be the *o-applicability domain*, or the *o-convergence domain*, of the method (X, F, x') and is denoted by $(X, F, x')^o$.

The necessary and sufficient conditions in order that the applicability and the b-applicability domains of an operator (X, F) include T^o, T^c, T^b and T, T^o, T^c and T^b denoting the sets of all null-convergent, convergent and bounded sequences, respectively, were quoted in [1], Section 2. In Section 3 of the same paper we have discussed the convergence domain and the o-convergence domain of a method (X, F, x').

Summary. In this paper we consider some special classes of the above mentioned function transformations of sequences. So, in Section 1 we introduce the notion of the operator (X, F) of the *type C* and give necessary and sufficient conditions under which the applicability and the b-applicability domains of such an operator include the sets T^o, T^c, T^b an T. Such an operator is a generalization of the notion of a matrix operator. The results obtained, except in the case of the set T, are some generalizations of the results known concerning the matrix operators.

In Section 2 we propose the definition of the *continuous* (X, F, x')-method and prove results containing conditions under which the convergence and the o-convergence domains of such a method (X, F, x') include the sets T^o, T^c, T^b and T. This notion is a generalization of Orlicz's definition of the continuous method (where $X=[x_0, x'))$ [5] (consequently, of Włodarski's continuous method, too [4]). Our results are either quite new (as in the case of T) or some generalizations and complements of the corresponding theorems for continuous methods of Orlicz and Włodarski, as well as of those for matrix methods.

1. Operators (X, F) of the C type

Definition 1.1. The operator (X, F) is said to be of the *type C* if there exists an at most countable family of topological spaces (X_i, θ_i) $(i=0, 1, 2, \ldots)$ such that: 1° $\bigcup_i X_i = X$; 2° for every $i=0, 1, 2, \ldots$ (X_i, θ_i) is a compact Hausdorff space with a finite or countable base; for each $i=0, 1, 2, \ldots$ and $j=0, 1, 2, \ldots$ the function $f_j|_{X_i}$ is continuous on (X_i, θ_i); 4° for any sequence $t=(t_j)$ the series $(x, F)(t)=\sum_{j=0}^{\infty} f_j(x)\, t_j$ converges uniformly on each set of the family X_i $(i=0, 1, 2, \ldots)$ whenever it converges on X.

Remark 1.1. Every matrix operator is of the type C.

Indeed, the matrix operator (F) defined by a matrix $F=(f_{ij})$ is associated with the set $X=\{0, 1, \ldots, i, \ldots\}$ and with the sequence $F=(f_j)$ of functions $f_j=f_j(x)$ $(j=0, 1, 2, \ldots)$ such that

$$f_j(i)=f_{ij} \quad (i=0, 1, 2, \ldots).$$

It is easy to verify that the choice $X_i=\{i\}$ $(i=0, 1, 2, \ldots)$ and the topologies θ_i $(i=0, 1, 2, \ldots)$ induced by the topology for the real (complex) numbers satisfy the conditions 1°—4° in Definition 1.1. Morever, as X_i $(i=0, 1, 2, \ldots)$ we can take any finite subsets of $X=\{0, 1, 2, \ldots\}$ satisfying 1° in Definition 1.1.

Remark 1.2. The power operators compose a subclass ot the class of C type operators (X, F), too.

Namely, the operator associated with a power series $s(x)=\sum_{j=0}^{\infty} a_j x^j$ can be considered as defined by the set $X=[0, R)$ and $F=\left(\frac{a_j x^j}{s(x)}\right)$, R being the convergence radius of the series in question. In this case we can take $X_i=[0, x_i]$ $(i=0, 1, 2, \ldots)$, assuming that (x_i) is a sequence of points of $[0, R)$ converging to R. Then the conditions 1°—3° of Definition 1.1 are obviously satisfied, while 4° follows from the well-known characteristic of the power series. In fact, more generally, we can take that X_i $(i=0, 1, 2, \ldots)$ is a family of compact subsets of $[0, R)$ covering it (and consequently, $X_i=[x_i, x_{i+1}]$, where $x_0=0$, too).

Remark 1.3. If an operator (X, F) is of the type C, then $t=(t_j)\in(X, F)^d$ implies $t=(t_j)\in(X_i, F)^b$ for every $i=0, 1, 2, \ldots$.

Indeed, since (X, F) is of the type C, the assumption $t=(t_j)\in(X, F)^d$ implies the uniform convergence of the series $(x, F)(t)=\sum_{j=0}^{\infty} f_j(x)\,t_j$ on X_i for each $i=0, 1, 2, \ldots$. Hence the function $(X, F)(t)$ in continuous on (X_i, θ_i) for every $i=0, 1, 2, \ldots$ and therefore bounded on X_i for each $i=0, 1, 2, \ldots$ (consequently, bounded on any union of finitely many sets of the family X_i $(i=0, 1, 2, \ldots)$, too).

To simplify the statements, we shall deal with a fixed (but arbitrarily chosen) covering X_i $(i=0, 1, 2, \ldots)$ of X in the sense of Definition 1.1.

Theorem 1.1. *The applicability domain $(X, F)^d$ of an operator (X, F) of the type C includes $T^o(T^c, T^b)$ if and only if the following condition is satisfied:*

(1.1) *the series $\sum_{j=0}^{\infty} |f_j(x)|$ converges uniformly on X_i for each $i=0, 1, 2, \ldots$.*

Proof. Obviously, the condition (1.1) implies $(T^o\subseteq T^c\subseteq)\ T^b\subseteq(X, F)^d$. To prove the contrary, assume $T^o\subseteq(X, F)^d$ and consider any set X_{i_0} of the given family X_i $(i=0, 1, 2, \ldots)$. By definition of the operator (X, F) of the type C, there exists a topology θ_{i_0} such that (X_{i_0}, θ_{i_0}) is a compact Hausdorff space with an at most countable base and such that each function $f_j(x)$ of the sequence $F=(f_j)$ is continuous on the space (X_{i_0}, θ_{i_0}). Moreover, $T^o\subseteq(X, F)^d$, as well as the supposition that (X, F) is of the type C, imply uniform convergence on X_{i_0} of the series $(x, F)(t)=\sum_{j=0}^{\infty} f_j(x)\,t_j$ for every bounded sequence $t=(t_j)$. Now, the condition (1.1) follows from

Lemma. Let Y be a compact Hausdorff space with a finite or countable neighbourhood base and let $f_j(x)$ $(j=0, 1, 2, \ldots)$ be a sequence of real (complex)-valued functions defined and continuous on Y. Then the following equivalence is valid:

The series $(x, F)(t)=\sum_{j=0}^{\infty} f_j(x)\,t_j$ converges uniformly on Y for every bounded sequence $t=(t_j)$ if and only if the series $\sum_{j=0}^{\infty} |f_j(x)|$ converges uniformly on Y.

This lemma is a natural generalization of Włodarski's result ([2], Lemma 3), whose scheme of the proof can remain unchanged.

Remark 1.4. When an operator (X, F) is of the type C, then the inclusion $T^o \subseteq (X, F)^d$ implies continuity on (X_i, θ_i) of the function defined by the series $\sum_{j=0}^{\infty} |f_j(x)|$, for every $i=0, 1, 2, \ldots$. (Hence this function is bounded on each set of the family X_i $(i=0, 1, 2, \ldots)$ (accordingly, bounded on the union of any finite subfamily of the family in question).) Similarly, under the given assumption, the series $\sum_{j=0}^{\infty} |f_j(x)|$ converges uniformly on any union of finitely many sets of the covering X_i $(i=0, 1, 2, \ldots)$. However, Abel's operator $\left(\text{i.e. the operator associated with the series } \sum_{j=0}^{\infty} x^j\right)$ shows that these assumptions do not imply the uniform convergence of the mentioned series on the union of all sets of the family X_i $(i=0, 1, 2, \ldots)$, that is on the set X.

Theorem 1.2. *The condition*:

(1.2) *for every* $i=0, 1, 2, \ldots$ *there exists an index* j_i *such that* $j>j_i \Rightarrow f_j(x) \equiv 0$ *on* X_i

is necessary and sufficient in order that the applicability domain of an operator (X, F) *of the type* C *includes the set* T *of all sequences.*

Proof. The sufficiency (1.2) for $(X, F)^d = T$ is obvious. To prove its necessity, suppose that X_{i_0} is an arbitrary set of the covering X_i $(i=0, 1, 2 \ldots)$. Then the assumption $(X, F)^d = T$ and the remark 1.3 imply $(X_{i_0}, F)^b = T$. Now the existence of an index j_{i_0} such that the function $f_j(x)$ vanishes on X_{i_0} for every $j>j_{i_0}$ follows from [1], Theorem 2.5.

Remark 1.4. The normal matrix operators show that $(X, F)^d = T$ does not imply the existence of an index j_0 with the property: $j>j_0 \Rightarrow f_j(x) \equiv 0$ on X (assuming the operator (X, F) is of the type C).

Theorem 1.3. *The b-applicability domain of an operator* (X, F) *of the type* C *includes* T^o (T^c, T^b) *if and only if* (1.1) *and the following condition are satisfied*:

(1.3) *there exists an index* i_0 *such that* $\sup_{\substack{x \in \cup X_i \\ i \geq i_0}} \sum_{j=0}^{\infty} |f_j(x)| < \infty.$

Proof. By Theorem 1.1, the inclusion $T^o \subseteq (X, F)^b$ implies (1.1). The condition (1.3) is a consequence of [1], the theorem 2.4 (whereby for i_0 we can take any of $i=0, 1, 2, \ldots$). Validity of the assertion that (1.1) and (1.3) imply $T^b \subseteq (X, F)^b$ follows from [1], Theorem 2.4 and Remark 1.3.

Remark 1.5. The conjunction of the requirements (1.1) and (1.3) is a stronger condition than $\sup_{x \in X} \sum_{j=0}^{\infty} |f_j(x)| < \infty$ (see [1], § 2, (3)), as we see from the example of the operator (X, F) with $X=[-3, -1]\cup\{0, 1, 2, \ldots\}$ and $F=(f_j)$:

$$f_j(x)=\begin{cases}\dfrac{(x+2)^2}{[1+(x+2)^2]^{j+1}}, & x\in[-3, -1]\\ \dfrac{1}{2^{x+j}}, & x\in\{0, 1, 2, \ldots\}\end{cases} \qquad (j=0, 1, 2, \ldots).$$

Theorem 1.4. *Necessary and sufficient condition in order that the b-applicability domain of an operator (X, F) of the type C includes all the sequences is that the following condition is satisfied:*

(1.4) *there exist indices j_0 and i_0 with the properties:*

$$1^\circ\ j>j_0 \Rightarrow f_j(x)\equiv 0 \text{ on } X;\ 2^\circ \sup_{\substack{x\in\cup X_i\\ i\geq i_0}} |f_j(x)|<\infty\ (j=0, 1, \ldots, j_0).$$

Proof. The necessity of (1.4) (where i_0 can be any of $i=0, 1, 2, \ldots$) follows from [1], Theorem 2.5. Its sufficiency results from the same theorem and Remark 1.3.

2. Continuous methods (X, F, x')

Definition 2.1. Let X be located in some Hausdorff topological space (X, θ). The method (X, F, x') is said to be *continuous* if there exists a neighbourhood base $\{O_i\}_{i=0,1,2,\ldots}$ at x' such that the corresponding operator (X, F) is of the C type with the choice $X_i=X\setminus O_i$ and $\theta_i=\theta$ $(i=0, 1, 2, \ldots)$ (see Definition 1.1).

Remark 2.1. If a method (X, F, x') is continuous, then the series $(x, F)(t)=\sum_{j=0}^{\infty} f_j(x)\,t_j$ converges uniformly on $X\setminus O$ for every neighbourhood O of x' (i.e. the same series is almost uniformly convergent on the set X with respect to the point x'), whenever it converges on X.

Remark 2.2. It follows from Definition 2.1 that the functions $f_j(x)$ $(j=0, 1, 2, \ldots)$ are continuous on $(X\setminus O, \theta)$ for every neighbourhood O of x'. Moreover, it can be shown that the same functions are continuous on (X, θ).

Remark 2.3. If the starting space (Y, θ) possesses an at most countable base, we can suppose that the neighbourhoods O_i $(i=0, 1, 2, \ldots)$ form a decreasing sequence of sets.

Remark 2.4. Every matrix method is continuous.

Indeed, in this case we have $X=\{0, 1, 2, \ldots, i, \ldots\}$, $x'=\infty$ and $F=(f_j)$ defined by $f_j(i)=f_{ij}$ $(i, j=0, 1, 2, \ldots)$, where $F=(f_{ij})$ is a given matrix. Obviously, then any neighbourhood base $\{O_i\}_{i=0,1,2,\ldots}$ at x' satisfies the requirement from Definition 2.1.

Remark 2.5. L. Włodarski [4] had studied the methods of the type (X, F, x') with $X=[x_0, x')$ and he had proposed the following definition:

A method $([x_0, x'), F, x')$ is said to be *continuous* if: 1° the functions $f_j(x)$ are continuous on $[x_0, x')$; 2° there exists an increasing sequence (x_n) of points of $[x_0, x')$ converging to x' such that for any sequence $t=(t_j)$ the series $(x, F)(t)=\sum_{j=0}^{\infty} f_j(x)\, t_j$ converges uniformly on $[x_{i-1}, x_i]$ whenever it converges at the points $x=x_{i-1}$ and $x=x_i$.

It is easy to see that a method $([x_0, x'), F, x')$ being continuous in the sense of the mentioned Włodarski's definition, W-continuous in the sequel, is also continuous in the sense of Definition 2.1. The neighbourhood base $O_i=(x_i, 2x'-x_i)$ when x' is finite and $O_i=(-\infty, 2x_0-x_i)\cup(x_i, +\infty)$ if $x'=\infty (i=0, 1, 2, \ldots)$ satisfies the condition given in Definition 2.1. However, it is clear that we can cite many other countable bases at x' with the same property.

Remark 2.6. W. Orlicz [5] proposed the following definition:

The method $([x_0, x'), F, x')$ is *continuous* if the condition 1° of the W-continuous method definition is satisfied as well as the following: 3° convergence of a series $(x, F)(t)=\sum_{j=0}^{\infty} f_j(x)\, t_j$ on $[x_0, x')$, $t=(t_j)$, implies its uniform convergence on $[x_0, x_1]$ for every $x_1<x'$.

If the method $([x_0, x'), F, x')$ is continuous in the sense of Orlicz's definition we shall call it O-continuous method. Obviously, an O-continuous method is continuous in the sense of Definition 2.1. Similarly as in the case of the W-continuous methods, a countable neighbourhood base at x' can be realized in many different ways. Also, observe that the above mentioned definitions of Włodarski and Orlicz were our inspiration for Definitions 1.1 and 2.1.

Theorem 2.1. *The convergence (o-convergence) domain of a continuous method (X, F, x') includes all null-convergent sequences if and only if the following conditions are satisfied:*

$$\text{(2.1)}\qquad \begin{cases}\text{the series } \sum_{j=0}^{\infty} |f_j(x)| \text{ converges uniformly on } X\setminus O \text{ for every neighbour-}\\ \text{hood } O \text{ of } x';\end{cases}$$

$$\text{(2.2)}\qquad \Big\{\text{there exists a neighbourhood } O_* \text{ of } x' \text{ such that } \sup_{x\in X\cap O_*} \sum_{j=0}^{\infty} |f_j(x)|<\infty$$

and

$$\text{(2.3)}\qquad \text{there exists } \lim_{x\to x'} f_k(x)=a_k\ (=0)\ (k=0, 1, 2, \ldots).$$

Observe that (2.1) and (2.2) imply $\sup_{x\in X} \sum_{j=0}^{\infty} |f_j(x)|<\infty$, which has been, along with the condition (2.3), given in [4] as a necessary and sufficient condition in order that the convergence (o-convergence) domain of a W-continuous method $([x_0, x'), F, x')$ includes T^o. Theorem 2.1 shows that then (as in general, also) we need a stronger condition: the conjunction of (2.1) and (2.2). The similar remark is valid in the case of the following theorem, too. This fact can be used in researching whether a given method (X, F, x') is continuous, in a way of the following example.

Precisely, we shall show that the method (X, F, x') with X and F from the remark 1.5 and $x' = \infty$ is not continuous. Namely, if the considered method is continuous, then the series $\sum_{j=0}^{\infty} |f_j(x)|$ will be convergent uniformly on $[-3, -1]$ (since $T^o \subseteq (X, F, x')^c$). However, this is not the case and therefore this method is not continuous. (Hence, for every neighbourhood base $\{O_i\}_{i=0,1,2,\dots}$ at x' there exist an index i_0 and a sequence $t = (t_j)$ such that the series $(x, F)(t) = \sum_{j=0}^{\infty} f_j(x)\, t_j$ converges on X, but this convergence is not uniform on $X \setminus O_{i_0}$.)

Proof of Theorem 2.1. From [1], Theorem 3.3, it follows that (2.1), (2.2) and (2.3) imply $T^o \subseteq (X, F, x')^c$. By the same theorem, $T^o \subseteq (X, F, x')^c$ implies the conditions (2.2) and (2.3). The condition (2.1) is obtained by the continuous method (X, F, x') definition and Theorem 1.1.

Theorem 2.2. *Let a method (X, F, x') be continuous. In order that every convergent sequence is (X, F, x')-convergent ((X, F, x')-o-convergent) it is necessary and sufficient that* (2.1), (2.2), (2.3) *as well as the following condition are satisfied:*

(2.4) *there exists* $\lim_{x \to x'} \sum_{j=0}^{\infty} f_j(x) = a\ (=0)$.

Theorem 2.3. *The conjunction of the conditions* (2.1), (2.3) *and*

(2.5) $$\lim_{x \to x'} \sum_{j=0}^{\infty} |f_j(x) - a_j| = 0$$

is necessary and sufficient in order that the convergence domain of a continuous method (X, F, x') includes all bounded sequences. (In the case of the o-convergence domain these conditions reduce to (2.1) *and* $\lim_{x \to x'} \sum_{j=0}^{\infty} |f_j(x)| = 0$.)

Proof. The conditions (2.1), (2.3) and (2.5) imply (2.2) (see [1], Theorem 3.5. i). Now the sufficiency of the mentioned conjunction and the necessity of (2.5) follow from [1], Theorem 3.5. The conditions (2.1) and (2.3) are necessary according to Theorem 2.1.

Another combination of conditions whose conjunction is equivalent to that from Theorem 2.3 can be given. More precisely, for the continuous methods (X, F, x') the analogy of the well-known theorem of Schur [7] is valid:

Theorem 2.4. *If a method (X, F, x') is continuous, then every bounded sequence is (X, F, x')-convergent if and only if the conditions* (2.3) *and the following are satisfied:*

(2.6) *the series* $\sum_{j=0}^{\infty} |f_j(x)|$ *is uniformly convergent on X.*

The proof of this theorem can be realized by the scheme of the proof of Theorem 3 in [6].

Theorem 2.5. *The convergence (o-convergence) domain of a continuous method (X, F, x') coincides with the set of all sequences if and only if the following condition is satisfied:*

$$(2.7)\quad \begin{array}{l} \textit{there exists an index } j_0 \textit{ such that: } 1'\ f_j(x)\equiv 0 \textit{ on } X \textit{ for every } j>j_0;\\ 2'\ \textit{there exist } \lim\limits_{x\to x'} f_j(x)=a_j\ (=0)\ (j=0, 1, \dots, j_0). \end{array}$$

Proof. The implication $(2.7) \Rightarrow (X, F, x')^c=T$ follows from [1], Theorem 3.6 (in fact, it is obvious). To prove the contrary, observe that in the case of a continuous method (X, F, x') and the corresponding operator (X, F) we have

$$(X, F, x')^o\subseteq(X, F, x')^c\subseteq(X, F)^b\subseteq(X, F)^d.$$

(All except $(X, F, x')^c\subseteq(X, F)^b$ is evident. However, $t=(t_j)\in(X, F, x')^c$ implies the existence of a neighbourhood O of x' with the property $\sup\limits_{x\in X\cap O} |(x, F)(t)|<\infty$. Denote by O_{i_0} a neighbourhood from the base $\{O_i\}_{i=0, 1, 2, \dots}$ such that $O_{i_0}\subseteq O$. By the definition 2.1 and the remark 1.3, we have $\sup\limits_{x\in X\setminus O} |(x, F)(t)|<\infty$. Hence $\sup\limits_{x\in X} |(x, F)(t)|<\infty$, i.e. $t=(t_j)\in(X, F)^b$.) Consequently, $(X, F, x')^c=T$ implies $(X, F)^b=T$. Now validity of (2.7) results from Theorems 2.5 [1] and 2.1.

Remark 2.7. Theorems 3.6 [1] and 2.5 show a difference between the continuous and the arbitrary methods (X, F, x'). Namely, in the case of an arbitrary method (X, F, x') the equality $(X, F, x')^c=T$ implies existence of an index j_0 and a neighbourhood O of x' such that for each $j>j_0$ the function $f_j(x)$ vanishes on the set $X\cap O$ "only".

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