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FIXPOINTS OF DECREASING MAPPINGS OF ORDERED SETS[1)]

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0. Let $(O, \leqslant)$ be any ordered set; in general, the set is only partially ordered.

0:0. If $f: O \to O$ is any mapping of O into O, we design by $I\ (O, f)$ the set of all invariant points of O relative to f; i.e.

$$(0.0) \qquad I(O, f): \{x \mid x \in O \wedge fx = x\}.$$

0:1. An ordered set $(O, \leqslant)$ is said to be *right* [left] *conditionally complete* if every non void subset M of O which is bounded from above [below] determines its own supremum [infimum]. *Conditionally complete* means to be both left and right conditionally complete.

One proves easily that right conditional completeness, left conditional completeness and completeness are three properties which are pairwise equivalent.

0:2. *σ-completeness.* If, for every $X \subset O$ of cardinality $\leqslant \aleph_0$ there exist the elements sup X and inf X of O, the set $(O, \leqslant)$ is quoted as *σ-complete.*

0:3. There is an extensive literature concerning the set $I(O, f)$; one knows particularly that if f is monotone the cases where f is increasing and decreasing behave quite distinctly.

In what follows we shall indicate some sufficient conditions on $(O, \leqslant)$ and f implying the existence of at least one fixpoint of f (cf. $1:6^d$, $2)\nearrow$, 2:13, 2:16).

0:4. If a mapping $f: O \to X$ is decreasing, the mapping f^2 is increasing in $(O, \leqslant)$; therefore if $(O, \leqslant)$ is left complete the set $I(O; f^2)$ is also a left complete non empty subset of $(O; \leqslant)$ (cf. [3] theorem 1). Now, every point a satisfying $f^2 a = a$ is such that the mapping f permutes the points a, fa. Therefore it is interesting to have a procedure to get solutions of $f^2 a = a$. (cf. 3. 1, **3.** $\mathbf{1^d}$).

0:5. Sets O_f, O^f and $O(f)$. For any $f: O \to O$ it is natural to consider the following sets.

$$(0:5:1) \qquad O_f = \{x \mid x \in O \wedge fx \leqslant x\}$$

$$(0:5:2) \qquad O^f = \{x \mid x \in O \wedge fx \geqslant x\}$$

$$(0:5:3) \qquad X(f) = \{x \mid x \in O \wedge x \,||\, fx\}.$$

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0 : 6. *Some inertness (perseverance) conditions* In connexion with mappings f between ordered sets the following conditions arise also, expressing a kind of perseverance or inertness of the order relations $\leqslant, \geqslant$ with respect to the operations inf, sup.

(0 : 6 : 1) *Condition* (r) for $r \in \{\leqslant, \geqslant \|\}$. If a set A satisfies $f\,a\,r\,a$ for every $a \in A$ and if sup A (resp. inf A) exists, then also f (sup A) r sup A (resp. f inf A r inf A).

1. Some elementary propositions.

1 : 1. Lemma. $O = O_f \cup O(f) \cup O^f$; $O^f \cap (O_f \cup O^f) = \varnothing$.

1 : 2. Lemma. *If f is increasing or decreasing in $(O, \leqslant)$, then*

$$I((O, \leqslant), f) = O_f \cap O^f.$$

1 : 3. Lemma. *If f is increasing, then $fO_f \subset O_f$, $fO^f \subset O^f$.*

1.4. Lemma. *If f is decreasing, then $fO_f \subset O^f$, $fO^f \subset O_f$.*

1 : 4 : 1. Example. If $D_0 = \{\ldots, -2, -1, 1, 2, \ldots\}$ and $x \in D_0 \mapsto fx \mapsto$ $\mapsto x - \text{sgn } x$, then: $f | D_0$ is decreasing, $D_{0f} = N: = \{1, 2, \ldots\}$, inf $D_{0f} = 1 = i$, $D_0{}^f = -N: = \{\ldots, -2, -1\}$, $S = \sup D_0^f = -1$, $fD_0^f = \{2, 3, \ldots\} \neq D_{0f}$, $fD_{0f} = \{-2, -3, \ldots\} \neq D_0{}^f$, $fi < S < i < fS$. If $n \in N$ then the mapping $f^n | D_0$ has no fixpoint.

1 : 5. Lemma. *If f is increasing then $fO^f \subset O^f$; moreover if $S = \sup O^f$ exists, then either $fS \leqslant S$ or $fS \| S$. In particular if $S \in O^f$, then $S = fS$,* i.e. $S \in I(O; f)$.

As a matter of fact, if $S < fS$, then $fS \leqslant f(fS)$, i.e. fS would be a member of O^f greater than the supremum S of O^f — absurdity. On the other hand, if $S \in O^f$, then $S \leqslant fS$; the relation $S < fS$ being impossible one has $S = fS$, S is a fixpoint for $f: O \to O$.

1 : 5[d]. Lemma. *If f is increasing and if $i = \inf O_f$ exists, then $\neg(fi < i)$. In particular, if $i \in O_f$ then $fi = i$, i.e. i is a fixpoint.*

Suppose that i exists and that, on the contrary, $fi < i$; this relation would imply $ffi \leqslant fi$ saying that $fi \in O_f$; consequently, the infimum i of O_f would be greater than the member fi of O_f — absurdity.

1 : 6. Lemma. *If f is decreasing, then $fO_f \subset O^f$, $fO^f \subset O_f$; if moreover* $\sup O_f$: $= s$ *exists, then $\neg(s < fs)$. If $s \in O^f$, then $s = fs$.*

As a matter of fact, if $s < fs$, then $fs \geqslant ffs$; in other words, fs would be a member of O_f greater than the supremum s of O_f — absurdity. Now, let assume $s \in O^f$; thus $s \leqslant fs$; the inequality $s < fs$ being impossible as we just saw, one has necessarily $s = fs$ saying that s is a fixpoint.

1 : 6[d]. Lemma. *If f is decreasing and if* $\inf O^f$: $= I$ *exists, then $\neg(fI < I)$. If $I \in O_f$, then $I = fI$ and $I \in O^f$.*

Proof. If I exists and if, by supposition, $fI < I$, this relation would imply $ffI \geqslant fI$ meaning that $fI \in O^f$; consequently, the element fI of O^f would be less than the infimum I of O^f — absurdity. If incidentally $I \in O_f$ i.e. if $fI \leqslant I$, this relation jointly with the relation $\rceil(fI < I)$ would mean that $fI = I$, i.e. that I is a fixpoint of $f: O \to O$.

1:7. Lemma. *If non empty subsets S, T of $(O, \leqslant)$ are such that $S \leqslant T$ (i.e. $x \leqslant y$ for every $x \in S \wedge y \in T$) and $s := \sup S \in O$, $t := \inf T \in O$, then*

$$s \leqslant t.$$

Since $S \leqslant \dot{T}$ for every $\dot{T} \in T$, so $\sup S \leqslant T$ i.e.

(1:7:1) $s \leqslant \dot{T}$ by the definition of the supremum. For the same reason, the relation (1:7:1) implies $s \leqslant \inf \dot{T}$, i.e. $s \leqslant \inf T$, thus $s \leqslant t$.

1:8. Lemma. *The set O^f, for $f\downarrow$ is a left portion of $(O, \leqslant)$, i.e. if $x \leqslant y \in O^f$, then also $x \in O^f$.*

Dually, the set O_f for $f\downarrow$, is a right portion of $(O, \leqslant)$, i.e. if $x \geqslant y \in O_f$, then also $x \in O_f$.

As a matter of fact the relation $x \leqslant y \in O^f$ implies $fx \geqslant fy \geqslant y$, that joint to $x \leqslant y$ implies $x \leqslant fx$, i.e. $x \in O^f$; etc.

2. Theorem. *Let $(O, \leqslant)$ be a non empty right conditionally complete ordered set and f a decreasing selfmapping of $(O, \leqslant)$ such that for at least one member $x \in O$ we have*

(2:1) $x \leqslant fx \vee x \geqslant fx$, *i.e.*

$\rceil(\forall x \in O \quad x \| fx)$.

Let us assume that

(2:2) $f \sup = \inf f$,

(2:3) *Each point of O_f is comparable with each point of O^f*

(2:4) *If $S := \sup O^f \in O$ exists then $S \geqslant fS$; Then*

(2:5) *$O^f \leqslant O_f$ (i.e. if $fx \geqslant x \in O$ and $fy \leqslant y \in O$, then $x \leqslant y$);*

(2:6) *the points $S := \sup O^f$, $i := \inf O_f$ exist and satisfy*

(2:7) $fS = S = \inf O_f$.

(2:8) $S := \sup O^f = \inf O_f =: i$.

2:9. Proof. Let us prove (2:5). Let $O(f) = \{x \mid x \in O, x \| fx\}$, then by hypothesis (2:1) $O(f) \neq O$. If $x \in O^f$ then $fx \in O_f$ because $x \in O^f$ means that $x \leqslant fx$ and therefore $fx \geqslant ffx$, i.e. $fx \in O_f$. Also, if $y \in O_f$ then $fy \in O^f$.

Since $O_f \cup O^f = O \setminus O(f) \neq \emptyset$ one concludes that $O^f \neq \emptyset \neq O_f$. Now, if (2:5) were false, there would be some $x \in O^f$ and some $y \in O_f$ such that either $x \| y$ or $y < x$. The first case being excluded by the condition (2:3), one should have $y < x$. Therefore, $fy \geqslant fx$; this relation joint to $x \leqslant fx$, $y \geqslant fy$ would yield $x \leqslant fx \leqslant fy \leqslant y$. i.e. $x \leqslant y$, contradicting the assumption $y < x$.

2:10. The set O^f being bounded from above (by every member of O_f), and the set $(O, \leqslant)$ being right conditionally complete the supremum $S := \sup O^f$ exists. Moreover, by the condition (2:4) one has $S \geqslant fS$.

2:11. We claim that $S \leqslant fS$ and consequently $fS = S$ proving that S is a requested fixed point for f. Now, the relation $S = \sup O^f$ implies $fS = f \sup O^f =$ $=$ (by the condition (2:2)) $\inf fO^f$; since $fO^f \subset O_f$ the member fS as $\inf fO^f$ satisfies $fS \geqslant \inf O_f$: $= i$; in other words $fS \geqslant i$.

On the other hand, by (2:5) $O^f \leqslant O_f$; therefore (v. L. 1:7) $\sup O^f \leqslant \inf O_f$, i.e. $S \leqslant i$; this relation joint to the above relation $i \leqslant fS$ implies the requested relation $S \leqslant fS$.

2:12. Let us prove (2:8). Since $S = fS$, one has $S \in O_f \cap O^f$. Since $S \leqslant O_f$, we have

$$S \leqslant i: = \inf O_f. \tag{2:12:1}$$

On the other hand, the element i as $\inf O_f$ satisfies $i \leqslant x$ for every $x \in O_f$; specifying here $x = S$, one becomes $i \leqslant S$, what jointly with (2:12:1) yields the requested equality $S = i$.

2:13. Theorem. *Let* $(O, \leqslant)$ *be a non empty conditionally complete ordered set and* $f: O \to O$ *be decreasing and satisfies*: (2:2), (2:8) *and*

$$fS \leqslant S \vee fS \geqslant S, \tag{2:14}$$

then $S \in I(O; f)$, *in other words* $fS = S$.

2:15. Proof. 2:15:1. If $fS \leqslant S$, then the arguments considered in the section 2:12 yield $fS = S$.

2:15:2. Let us now consider the case $fS \geqslant S$. Now, by (2:8), $S = i$ and consequently $fS = fi = f \inf O_f =$ (by (2:2)) $= \sup fO_f \leqslant$ (because $fO_f \subset O^f$) $\leqslant \sup O^f = S$. Thus $fS \leqslant S$, that joint to $fS \geqslant S$ yields $fS = S$. This completes the proof of the theorem 2:13.

2:16. Theorem. *Let* $(O, \leqslant)$ *be a non empty conditionally complete ordered set and* $f: (O, \leqslant) \to (O; \leqslant)$ *be decreasing satisfying* (2:2) *and such that there are non empty sets* X, Y *such that*

$$X \subset O^f,\ Y \subset O_f,\ fX \subset Y,\ fY \subset X \tag{2:17}$$

$$\sup X = \inf Y = : z \text{ and} \tag{2:18}$$

$$fz \leqslant z \vee fz \geqslant z; \tag{2:19}$$

then

$$fz = z, \text{ i.e. } z \in I(O; f). \tag{2:20}$$

Proof. 2:21. *Case* $fz \leqslant z$. Let us prove that also $fz \geqslant z$. Now, $z = \sup X$ implies $fz = \inf fX \geqslant$ (because by (2:17) $fX \subset Y$) $\geqslant \inf Y = z$.

2:22. *Case* $fz \geqslant z$. The equality $z = \inf Y$ in (2:18) implies by (2:2) $fz = \sup fY \leqslant \sup X = z$ (because $fY \subset O_f$, by (2:18)); thus $fz \leqslant z$. Consequently, in either case we have $fz = z$.

2:23. Remark. For the case $X = O^f$, $Y = O_f$, the theorem 2:16 yields the statement 2:13.

2:24. The statement 2:16 is more general than the statement 2:13. This follows of the following.

2:25. Example. Let M be any set of at least two members; R being the ordered set of real numbers, let us consider the ordered set $(O, \prec)$, where $O = M \times R$ and for

$$(x, y),\ (x', y') \in M \times R \text{ one has}$$
$$(x, y) \prec (x', y') \Leftrightarrow x = x' \wedge y \leq y'; \text{ in particular,}$$
$$(x, y) \| (x', y') \Leftrightarrow x \neq x'.$$

Let $f | O$ be defined by $f(x, y) = (x, -y)$; then

$$O_f = \{(x, y) | (x, y) \in O \wedge y \geq 0\},\ O^f = \{(x, y) | (x, y) \in O \wedge y \leq 0\},$$
$$I(O, f) = \{(x, 0) | x \in M\};$$

obviously, neither $\sup O^f$ nor $\inf O_f$ exists; therefore, the theorem 2:13 does not work for the set $(M \times R; \prec)$; on the contrary, the theorem 2:16 does work: it suffices to put for $m \in M$:

$$X = X_m = \{(m, y) | y \in R(-\infty, 0],\ Y = Y_m = \{(m\ y) | y \in R[0, +\infty),$$
$$z = z_m = (m, 0).$$

3. A way to get some solution of $fa^2 = a$.

3.1. Lemma. *Assumptions. Let f be decreasing and let f satisfy* (2:2); *let $(O, \leq)$ be σ-complete and the conditions* $(\leq), (\geq)$ *of* (0:5:1) *be satisfied. Then for every $a \in O^f$ the element s of O defined by*

$$(3:2) \qquad s = s(a) = \sup\{f^{(0)} a = a,\ f^2 a,\ f^4 a, \ldots\}^{1)}$$

exists, the sequence

$$(3:3) \qquad f^{(2k)} s \qquad (k = 0, 1, 2, \ldots)$$

is a decreasing sequence of members from O^f such that

$$(3:4) \qquad I := \inf f^{(2k)} s \in O^f; \qquad (k \in I_\omega);$$

the sequence

$$(3:5) \qquad f^{(2k+1)} s \qquad (k \in I_\omega)$$

is an increasing sequence of elements of O_f, and

$$(3:6) \qquad S = \sup_{k \in N} f^{(2k+1)} s \in O_f;$$

one has

$$(3:7) \qquad fI = S,\quad fS = I,\quad f^2 I = I,\quad f^2 S = S$$

i.e. f permutes I, S and $\{I, S\} \subset I(f^2, O)$.

Proof. The existence of s, S and I (depending on a) is implied by the σ-completeness $(O, \leq)$. Since $f^{2k} a \in O^f$, the condition $(\leq)$ implies $s \leq fs$, and therefore $fs \geq f^2 s$, i.e. $fs \in O_f$. Since $s \geq f^{2k} a$ for every $k \in I\omega = \{0, 1, 2, \ldots\}$, one has $fs \leq f^{2k+1} a \in O_f$.

If $s = fs$, then $s = S = I$ and all is proved. Therefore, let us consider the case $s < fs$; then $a < fa$; from (3:2) and (2:2) we deduce that

$$(3:8) \qquad fs = \inf f^{2k+1} a := i$$

[1]) In general, neither the set $\{a, f^2 a, f^4 a, \ldots\}$ nor the set $\{fa f^3 a. f^5, ,..\}$ is a chain.

and thus $s < f^{2k+1}a$; in virtue of (3.2) again one implies that $f^{2h}a < f^{2k+1}a$ for every $k \in I_\omega$. From (3.8) we get $f^2 s = \sup f^{2k+2} a \leqslant \sup_k f^{2k} a := s$; therefore in particular $f^2 s \leqslant s$ and $fs \geqslant f^2 s \leqslant s$; thus, acting by $f^2 : f^3 s \geqslant f^4{}_s \leqslant f^2 s$. Consequently

$$(3:9) \qquad f^4 s \leqslant f^2 s \leqslant s.$$

From (3 : 10) applying f^2 stepwise to (3.9) one proves that the sequence (3 : 3) is decreasing, its members are in O^f, so also is I from (3 : 4). Since (3 : 3) is decreasing, the sequence (3 : 5) is increasing; its elements belong to O_f; therefore, by hypothesis, (3 : 6) is holding and $fS \leqslant S$. Let us still prove (3 : 7). We have $fI = f\ (\inf f^{2k} s) =$ (by (2 : 2)) $= \sup f^{2k+1} s = S$, thus $fI = S$. From there

$$f^2 I = fS = f \sup F^{2k+1} s = \inf f^{2k+2} s = I,$$

the sequence (3 : 4) being decreasing. This proves 3 : 1.

3.1^d. Lemma. *Assumptions are like those in L. 3.1. Then for every $a \in O_f$ the element i of O defined by*

$$i : \inf \{f^{(0)} a = a,\ f^2 a,\ f^4 a, \ldots\} \in O$$

exists; the corresponding sequence of members

$$f^{(2k)} i \in O_f, \qquad (k = 0, 1, 2, \ldots)$$

is increasing and we have the element

$$\sup f^{(2k)} i =: S \in O_f;$$

the sequence

$$f^{(2k+1)} i \qquad (k = 0, 1, \ldots)$$

is decreasing, its members belong to O_f and

$$I = \inf f^{(2k+1)} i.$$

One has

$$fI = S, \quad fS = I.$$

i.e. *f permutes S, I; and $\{I, S\} \subset I(f^2, O)$.*

The proof of 3 : 1^d runs like that one of L. 3 : 1.

3.2. Remark. If $a \| fa$, we know no procedure starting with a and yielding a fixed point of f or of f^2. In general, if $O = O(f)$, there is no fixed point of f.

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[1] The following corrections in this paper should be made:
p. 168^9 decreasing → increasing
p. 169_{11} should be dropped
p. 169_7 than → then; 170_5 $(\cdot, 1) \to (\cdot, 1]$
171_{18} $\leqslant \to \geqslant$.
173^{11} 1 : 2^d should be dropped.