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ON (m, n)-REGULAR SEMIGROUPS

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Let S be a semigroup, m, n non-negative integers. We say that S is (m, n)-regular if for every element $a \in S$ there exists an $x \in S$ such that $a = a^m x a^n$ (a^0 is defined as an operator element, so that $a^0 x = x a^0 = x$). (S. Lajos [5]). R. Croisot [2] says that S has a condition (m, n).

S. Lajos [4] introduced the concept of (m, n)-ideals. The subsemigroup A of semigroup S is an (m, n)-ideal if A satisfies the relation

$$A^m S A^n \subseteq A$$

where m, n are non-negative integers. (Here A^0 is defined as an operator element, so that $A^0 S = S A^0 = S$). The principal (m, n)-ideal, generated by the element a, is

$$[a]_{(m, n)} = \bigcup_{i=1}^{m+n} \{a^i\} \cup a^m S a^n.$$

K. Iséki [3] gave ideal-characterization of regular semigroups. S. Lajos [6] gave bi-ideal characterization of regular semigroups. In this note, the (m, n)-ideal characterization of (m, n)-regular semigroups will be given. For $m = n = 1$ we obtain K. Iséki's theorem (Theorem 3) and S. Lajos's theorem (Theorem 2).

Lemma. *Let S be a semigroup, m, n positive integers, $[a]_{(m, n)}$ the principal (m, n)-ideal, generated by element a. Then*

1°. $([a]_{(m, 0)})^m S = a^m S$

2°. $S([a]_{(0, n)})^n = S a^n$

3°. $([a]_{(m, n)})^m S ([a]_{(m, n)})^n = a^m S a^n$.

Proof 1°. As $[a]_{(m, 0)} = \bigcup_{i=1}^{m} \{a^i\} \cup a^m S$, we have

$$([a]_{(m, 0)})^m S = \left(\bigcup_{i=1}^{m} \{a^i\} \cup a^m S\right)^m S = \left(\bigcup_{i=1}^{m} \{a^i\} \cup a^m S\right)^{m-1}.$$

$$\left(\bigcup_{i=1}^{m} \{a^i\} \cup a^m S\right) S = \left(\bigcup_{i=1}^{m} \{a^i\} \cup a^m S\right)^{m-1} \left(\bigcup_{i=1}^{m} a^i S \cup a^m SS\right) =$$

$$=\left(\bigcup_{i=1}^{m}\{a^i\}\cup a^m S\right)^{m-1} aS=\left(\bigcup_{i=1}^{m}\{a^i\}\cup a^m S\right)^{m-2}\left(\bigcup_{i=1}^{m}\{a^i\}\cup a^m S\right)aS=$$

$$=\left(\bigcup_{i=1}^{m}\{a^i\}\cup a^m S\right)^{m-2}\left(\bigcup_{i=1}^{m} a^{i+1}S\cup a^m SaS\right)=\left(\bigcup_{i=1}^{m}\{a^i\}\cup a^m S\right)^{m-2} a^2 S=$$

$$\cdots=\left(\bigcup_{i=1}^{m}\{a^i\}\cup a^m S\right)^{m-(m-1)} a^{m-1} S=\left(\bigcup_{i=1}^{m}\{a^i\}\cup a^m S\right)a^{m-1} S=$$

$$=\bigcup_{i=1}^{m} a^{i+m-1} S\cup a^m Sa^{m-1} S=a^m S.$$

Therefore,

$$([a]_{(m,0)})^m S=a^m S.$$

Analogously, we can prove 2°.

3°. $$([a]_{(m,n)})^m S=\left(\bigcup_{i=1}^{m+n}\{a^i\}\cup a^m Sa^n\right)^m S=\left(\bigcup_{i=1}^{m+n}\{a^i\}\cup a^m Sa^n\right)^{m-1}\cdot$$

$$\left(\bigcup_{i=1}^{m+n}\{a^i\}\cup a^m Sa^n\right)S=\left(\bigcup_{i=1}^{m+n}\{a^i\}\cup a^m Sa^n\right)^{m-1}\left(\bigcup_{i=1}^{m+n} a^i S\cup a^m Sa^n S\right)=$$

$$=\left(\bigcup_{i=1}^{m+n}\{a^i\}\cup a^m Sa^n\right)^{m-1} aS=\cdots=a^m S.$$

In the same way,

$$S([a]_{(m,n)})^n=Sa^n.$$

Therefore,

$$([a]_{(m,n)})^m S([a]_{(m,n)})^n=a^m S([a]_{(m,n)})^n=a^m Sa^n.$$

Theorem 1. *Let S be a semigroup, m, n positive integers, $\mathcal{R}_{(m,0)}$ the set of all $(m, 0)$-ideals and $\mathcal{L}_{(0,n)}$ the set of all $(0, n)$-ideals of S. Then,*

1°. *S is $(m, 0)$-regular* $\Leftrightarrow (\forall R\in\mathcal{R}_{(m,0)})\quad (R=R^m S)$.

2°. *S is $(0, n)$-regular* $\Leftrightarrow (\forall L\in\mathcal{L}_{(0,n)})\quad (L=SL^n)$.

Proof. Let S be a (m, o)-regular semigroup, i.e.

(1) $$(\forall a\in S)(\exists x\in S)(a=a^m x)$$

and R a $(m, 0)$-ideal of S. According to the definition of $(m, 0)$-ideal, we have $R^m S\subset R$. Let $a\in R$. From (1), $a=a^m s\in R^m S$, i.e. $R\subset R^m S$. Hence, $R=R^m S$. Conversely, let for semigroup S

(2) $$(\forall R\in\mathcal{R}_{(m,0)})(R=R^m S).$$

For $R=[a]_{(m,0)}$, we obtain from (2)

$$[a]_{(m,0)}=([a]_{(m,0)})^m S.$$

According to Lemma, we have

$$[a]_{(m,0)}=a^m S.$$

As $a\in[a]_{(m,0)}$, we have $a\in a^m S$, i.e. S is $(m, 0)$-regular.

Analogously, we can prove 2°.

Theorem 2. *Let S be a semigroup, m, n non-negative integers, $\mathcal{A}_{(m,n)}$ the set of all (m, n)-ideals in S. Then*

(3) $$S \text{ is } (m, n)\text{-regular} \Leftrightarrow (\forall A \in \mathcal{A}_{(m,n)}) (A^m SA^n = A).$$

Proof. If $m=n=0$, (3) holds because S is the only $(0, 0)$-ideal. If $m=0$, $n\neq 0$ we obtain that

$$S \text{ is } (0, n)\text{-regular} \Leftrightarrow (\forall A \in \mathcal{A}_{(0,n)}) (SA^n = A).$$

This is true according to 2°, Theorem 1. Analogously, if $m\neq 0$, $n=0$, (3) holds. (Theorem 1, 1°).

$m\neq 0$ and $n\neq 0$. Let S be an (m, n)-regular semigroup and A a (m, n)-ideal. Then

(4) $$(\forall a \in S) (\exists x \in S) (a = a^m xa^n)$$

Let $a\in A$. From (4), $a=a^m sa^n$ i.e. $a\in A^m SA^n$. Hence, $A\subset A^m SA^n$. According to the definition of (m, n)-ideal, we have $A^m SA^n = A$.

Conversely, let $(\forall A \in \mathcal{A}_{(m,n)}) (A^m SA^n = A)$. If $a\in S$, then

$$[a]_{(m,n)} = ([a]_{(m,n)})^m S ([a]_{(m,n)})^n.$$

According to 3°, Lemma, we have

$$[a]_{(m,n)} = a^m Sa^n.$$

Therefore $a\in a^m Sa^n$, i.e. S is (m, n)-regular semigroup.

Theorem 3°. *Let S be a semigroup, m, n non-negative integers, $\mathcal{R}_{(m,0)}$ the set of all $(m, 0)$-ideals, $\mathcal{L}_{(0,n)}$ the set of all $(0, n)$-ideals in S. Then*

(5) $$S \text{ is } (m, n)\text{-regular} \Leftrightarrow (\forall R \in \mathcal{R}_{(m,0)}) (\forall L \in \mathcal{L}_{(0,n)}) (R\cap L = R^m L \cap RL^n)$$

where $R^0 L \overset{\text{def}}{=} L$ *and* $RL^0 \overset{\text{def}}{=} R$.

Proof. If $m=n=0$ (5) holds for every semigroup S i.e. every semigroup S is $(0, 0)$-regular. If $m=0$, $n\neq 0$, we have $R=S$. Then $R\cap L = R^m L\cap RL^n$ becomes $L=L\cap SL^n$ and from that $L\subset SL^n$. Hence, $L=SL^n$. Then (5) becomes

$$S \text{ is } (0, n)\text{-regular} \Leftrightarrow (\forall L \in \mathcal{L}_{(0,n)}) (L = SL^n).$$

This is true according to Theorem 1. Analogously, if $m\neq 0$, $n=0$, (5) gives that

$$S \text{ is } (m, 0)\text{-regular} \Leftrightarrow (\forall R \in \mathcal{R}_{(m,0)}) (R = R^m S).$$

This is true according to Theorem 1.

$m\neq 0$ and $n\neq 0$. For every semigroup S holds:

$$(R^m L \subset R^m S \subset R) \wedge (RL^n \subset SL^n \subset L)$$

where R is $(m, 0)$-ideal and L $(0, n)$-ideal. Hence,

(6) $$R^m L \cap RL^n \subset R \cap L.$$

Let S be (m, n)-regular semigroup i.e. holds (4). Let $a\in R\cap L$. Then, from (4) it follows $a=(a^m s) a^n \in RL^n$ and $a=a^m (sa^n) \in R^m L$, i.e. $a\in R^m L\cap RL^n$.

Therefore,

(7) $$R\cap L\subset R^m L\cap RL^n.$$

According to (6) and (7),

$$R\cap L=R^m L\cap RL^n.$$

Conversely, let for semigroup S be

(8) $$(\forall R\in\mathcal{R}_{(m,0)})\ (\forall L\in\mathcal{L}_{(0,n)})\ (R\cap L=R^m L\cap RL^n).$$

Then

(9) $$(\forall R\in\mathcal{R}_{(m,0)}\ (\forall L\in\mathcal{L}_{(0,n)}\ (R\cap L\subset RL).$$

For $R=[a]_{(m,0)}$, $L=S$, we obtain from (8) $[a]_{(m,0)}\subset([a]_{(m,0)})^m S$. According to Lemma, we have $[a]_{(m,0)}\subset a^m S$, i.e.

(10) $$[a]_{(m,0)}=a^m S.$$

For $R=S$, $L=[a]_{(0,n)}$, we obtain from (8), $[a]_{(0,n)}\subset S([a]_{(0,n)})^n$. According to Lemma, we have $[a]_{(0,n)}\subset Sa^n$, i.e.

(11) $$[a]_{(0,n)}=Sa^n.$$

As $a^m S$ is a right ideal (i.e. (1, 0)-ideal), $a^m S$ is $(m, 0)$-ideal. Analogously, Sa^n is $(0, n)$-ideal. Then, from (9), we obtain

$$a^m S\cap Sa^n\subset a^m SSa^n\subset a^m Sa^n.$$

According to (10) and (11), we have

$$[a]_{(m,0)}\cap[a]_{(0,n)}\subset a^m Sa^n$$

i.e. $a\in a^m Sa^n$, which means that S is (m, n)-regular semigroup.

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