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# ON THE CONVERGENCE OF CERTAIN SEQUENCES, V

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This note is concerned with the convergence of sequences $(x_n^1), \ldots, (x_n^k)$, defined by

$$x_{n+p}^i = f_i\left(x_n^1, \ldots, x_n^k, x_{n+1}^1, \ldots, x_{n+1}^k, \ldots, x_{n+p-1}^1, \ldots, x_{n+p-1}^k\right) \tag{1}$$

$(i=1, \ldots, k)$, where $x_j^i$ $(i=1, \ldots, k;\ j=1, \ldots, p)$ are arbitrary.

In further text we shall suppose that $(E, d)$ is a complete metric space, that $f_i : E^{pk} \to E$ and that $\mathbf{A}_1, \ldots, \mathbf{A}_p$ are real matrices with nonnegative entries. If $\mathbf{A}$ is a square matrix, $N(\mathbf{A})$ will denote any norm of $\mathbf{A}$, i.e. any nonnegative real functional with the properties

(i) $N(\mathbf{A}) = 0$ if and only if $\mathbf{A} = \mathbf{O}$

(ii) $N(\lambda \mathbf{A}) = \lambda N(\mathbf{A})$ $(\lambda \geqslant 0)$

(iii) $N(\mathbf{A}+\mathbf{B}) \leqslant N(\mathbf{A}) + N(\mathbf{B})$

(iv) $N(\mathbf{AB}) \leqslant N(\mathbf{A})\, N(\mathbf{B})$

(v) $\mathbf{A} \leqslant \mathbf{B}$ implies $N(\mathbf{A}) \leqslant N(\mathbf{B})$.

E. Udovičić proved in [1] that if

$$\left\| \begin{matrix} d\big(f_1(u_{11}, \ldots, u_{1k}, u_{21}, \ldots, u_{2k}, \ldots, u_{p1}, \ldots, u_{pk}), \\ \qquad f_1(u_{21}, \ldots, u_{2k}, u_{31}, \ldots, u_{3k}, \ldots, u_{p+1,1}, \ldots, u_{p+1,k})\big) \\ \vdots \\ d\big(f_k(u_{11}, \ldots, u_{1k}, u_{21}, \ldots, u_{2k}, \ldots, u_{p1}, \ldots, u_{pk}), \\ \qquad f_k(u_{21}, \ldots, u_{2k}, u_{31}, \ldots, u_{3k}, \ldots, u_{p+1,1}, \ldots, u_{p+1,k})\big) \end{matrix} \right\|$$

$$\leqslant \sum_{i=1}^{p} \mathbf{A}_i \left\| \begin{matrix} d(u_{i1}, u_{i+1,1}) \\ \vdots \\ d(u_{ik}, u_{i+1,k}) \end{matrix} \right\| \tag{2}$$

and

(3)
$$\sum_{i=1}^{p} N(\mathbf{A}_i) < 1,$$

then sequences $(x_n^1), \ldots, (x_n^k)$ are convergent in $E$, and the unique solution of the system

(4)
$$x^i = f_i(x^1, \ldots, x^k, x^1, \ldots, x^k, \ldots, x^1, \ldots, x^k) \quad (i = 1, \ldots, k)$$

is provided by
$$x^i = \lim_{n \to +\infty} x_n^i \qquad (i = 1, \ldots, k).$$

On the basis of some results given in [2], we have asked in [3] the following question:

Is it possible to replace condition (3) by the weaker condition
$$N\left(\sum_{i=1}^{p} \mathbf{A}_i\right) < 1?$$

This paper gives an affirmative answer to that question.

Theorem. *Let $(E, d)$ be a complete metric space and let the functions $f_i : E^{pk} \to E\,(i = 1, \ldots, k;$ $p$ a fixed positive integer) satisfy* (2) *for every $u_{11}, \ldots, u_{pk} \in E$, where*
$$N\left(\sum_{i=1}^{p} \mathbf{A}_i\right) \leqslant q < 1.$$

*Then the sequences $(x_n^1), \ldots, (x_n^k)$, defined by* (1) *converge to $x^1, \ldots, x^k$, respectively $(x^1, \ldots, x^k \in E)$, where $(x^1, \ldots, x^k)$ is the unique solution of the system* (4).

Proof. Let
$$\mathbf{D}_r = \begin{Vmatrix} d(x_r^1, x_{r+1}^1) \\ \vdots \\ d(x_r^k, x_{r+1}^k) \end{Vmatrix}.$$

From (2) follows

(5)
$$\mathbf{D}_{n+p+i} \leqslant \sum_{\nu=1}^{p} \mathbf{A}_\nu \mathbf{D}_{n+i+\nu-1}.$$

Adding inequalities (5) for $i = 0, 1, \ldots, m$, we get
$$\sum_{i=0}^{m} \mathbf{D}_{n+p+i} \leqslant \sum_{\nu=1}^{p} \mathbf{A}_\nu \sum_{i=0}^{m} \mathbf{D}_{n+i+\nu-1}$$
$$\leqslant \left(\sum_{\nu=1}^{p} A_\nu\right)\left(\sum_{i=0}^{m} \mathbf{D}_{n+p+i} + \sum_{i=0}^{p-1} \mathbf{D}_{n+i}\right).$$

Hence

$$N\left(\sum_{i=0}^{m} \mathbf{D}_{n+p+i}\right) \leqslant N\left(\sum_{\nu=1}^{p} \mathbf{A}_\nu\right) N\left(\sum_{i=0}^{m} \mathbf{D}_{n+p+i} + \sum_{i=0}^{p-1} \mathbf{D}_{n+i}\right)$$

$$\leqslant qN\left(\sum_{i=0}^{m} \mathbf{D}_{n+p+i}\right) + qN\left(\sum_{i=0}^{p-1} \mathbf{D}_{n+i}\right),$$

which implies

$$N\left(\sum_{i=0}^{m} \mathbf{D}_{n+p+i}\right) \leqslant \frac{q}{1-q} N\left(\sum_{i=1}^{p-1} \mathbf{D}_{n+i}\right). \tag{6}$$

Let $\mathbf{D} = \limsup_{n\to+\infty} \mathbf{D}_n$. Then from (5) we find $N(\mathbf{D}) \leqslant qN(\mathbf{D})$, i.e. $N(\mathbf{D}) = 0$, or equivalently $\lim_{n\to+\infty} \mathbf{D}_n = \mathbf{O}$, which together with (6) yields

$$\lim_{n\to+\infty} d(x^i_{n+p+m}, x^i_{n+p}) = 0 \qquad (i = 1, \dots, k).$$

Hence, the sequences $(x^1_n), \dots, (x^k_n)$ are Cauchy's and therefore convergent. Let $x^i = \lim_{n\to+\infty} x^i_n$. Then

$$\left\| \begin{matrix} d\left(x^1_{n+p}, f_1(x^1, \dots, x^k, \dots, x^1, \dots, x^k)\right) \\ \vdots \\ d\left(x^k_{n+p}, f_k(x^1, \dots, x^k, \dots, x^1, \dots, x^k)\right) \end{matrix} \right\|$$

$$= \left\| \begin{matrix} d\left(f_1(x^1_n, \dots, x^k_n, \dots, x^1_{n+p-1}, \dots, x^k_{n+p-1}), f_1(x^1, \dots, x^k, \dots, x^1, \dots, x^k)\right) \\ \vdots \\ d\left(f_k(x^1_n, \dots, x^k_n, \dots, x^1_{n+p-1}, \dots, x^k_{n+p-1}), f_k(x^1, \dots, x^k, \dots, x^1, \dots, x^k)\right) \end{matrix} \right\|$$

$$\leqslant \left\| \begin{matrix} d\big(f_1(x^1_n, \dots, x^k_n, \dots, x^1_{n+p-1}, \dots, x^k_{n+p-1}), \\ \qquad f_1(x^1_{n+1}, \dots, x^k_{n+1}, \dots, x^1, \dots, x^k)\big) \\ \vdots \\ d\big(f_k(x^1_n, \dots, x^k_n, \dots, x^1_{n+p-1}, \dots, x^k_{n+p-1}), \\ \qquad f_k(x^1_{n+1}, \dots, x^k_{n+1}, \dots, x^1, \dots, x^k)\big) \end{matrix} \right\|$$

$$+ \cdots +$$

$$\left\| \begin{matrix} d\left(f_1(x^1_{n+p-1}, \dots, x^k_{n+p-1}, \dots, x^1, \dots, x^k), f_1(x^1, \dots, x^k, \dots, x^1, \dots, x^k)\right) \\ \vdots \\ d\left(f_k(x^1_{n+p-1}, \dots, x^k_{n+p-1}, \dots, x^1, \dots, x^k), f_k(x^1, \dots, x^k, \dots, x^1, \dots, x^k)\right) \end{matrix} \right\|$$

$$\leqslant \left(\sum_{\nu=1}^{p} \mathbf{A}_\nu\right)\left(\left\| \begin{matrix} d(x^1_{n+p-1}, x^1) \\ \vdots \\ d(x^k_{n+p-1}, x^k) \end{matrix} \right\| + \sum_{i=0}^{p-2} \left\| \begin{matrix} d(x^1_{n+i}, x^1_{n+i+1}) \\ \vdots \\ d(x^k_{n+i}, x^k_{n+i+1}) \end{matrix} \right\|\right),$$

which implies

$$\lim_{n\to+\infty} d\big(x^i_{n+p}, f_i(x^1, \ldots, x^k, \ldots, x^1, \ldots, x^k)\big) = 0 \qquad (i = 1, \ldots, k)$$

or

$$\lim_{n\to+\infty} x^i_n = f_i(x^1, \ldots, x^k, \ldots, x^1, \ldots, x^k) \qquad (i = 1, \ldots, k).$$

Hence, $(x^1, \ldots, x^k)$ is a solution of the system (4). To prove that this solution is unique, suppose that $(y^1, \ldots, y^k)$ is also a solution of (4). Then

$$\left\| \begin{matrix} d(x^1, y^1) \\ \vdots \\ d(x^k, y^k) \end{matrix} \right\| =$$

$$= \left\| \begin{matrix} d(f_1(x^1, \ldots, x^k, \ldots, x^1, \ldots, x^k), f_1(y^1, \ldots, y^k, \ldots, y^1, \ldots, y^k)) \\ \vdots \\ d(f_k(x^1, \ldots, x^k, \ldots, x^1, \ldots, x^k), f_k(y^1, \ldots, y^k, \ldots, y^1, \ldots, y^k)) \end{matrix} \right\|$$

$$\leqslant \sum_{\nu=1}^{p} \mathbf{A}_\nu \left\| \begin{matrix} d(x^1, y^1) \\ \vdots \\ d(x^k, y^k) \end{matrix} \right\|.$$

Therefore,

$$N\left(\left\| \begin{matrix} d(x^1, y^1) \\ \vdots \\ d(x^k, y^k) \end{matrix} \right\|\right) \leqslant qN\left(\left\| \begin{matrix} d(x^1, y^1) \\ \vdots \\ d(x^k, y^k) \end{matrix} \right\|\right),$$

which is impossible.

The theorem is proved.

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