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NOTES ON PARTIAL DIFFERENTIAL EQUATIONS III: TWO DIRICHLET TYPE PROBLEMS FOR POISSON'S EQUATION

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1. Let **R** be the set of all real numbers and let **C** be the set of all complex numbers. Furthermore, let $K=\{(x, y)\,|\,x^2+y^2=R^2\}\subset \mathbf{R}^2$, $D=\operatorname{int} K=\{(x, y)\,|\,x^2+y^2<R^2\}$, and $D_0=D-\{0\}=\{(x, y)\,|\,0<x^2+y^2<R^2\}$.

This note is concerned with the two dimensional Poisson's equation (nonhomogeneous Laplace equation)

$$u_{xx}+u_{yy}=F(x, y) \tag{1}$$

where F is a given function, and with boundary problems of Dirichlet type, namely

$$u=f \quad \text{for} \quad (x, y)\in K \tag{2}$$

where f is a given function.

2. Poisson's equation (1) can be reduced to the Laplace equation

$$u_{xx}+u_{yy}=0 \tag{3}$$

provided that one particular solution of (1) is known.

In this section we shall therefore give a formal method for determining a particular solution of (1).

Together with equation (1) we consider an other equation of the same type, namely

$$v_{xx}+v_{yy}=G(x, y). \tag{4}$$

Multiply (4) by i and add it to (1). We get

$$w_{z\bar{z}}=H(z, \bar{z}), \tag{5}$$

where $z=x+iy$, $\bar{z}=x-iy$, $w(z, \bar{z})=u(x, y)+iv(x, y)$, $w_z=\frac{1}{2}((u_x+v_y)+i(v_x-u_y))$, $w_{\bar{z}}=\frac{1}{2}((u_x-v_y)+i(v_x+u_y))$, $H(z, \bar{z})=\frac{1}{4}(F(x, y)+iG(x, y))$.

A particular solution of (5) is readily seen to be

$$w(z, \bar{z}) = \int \left(\int H(z, \bar{z})\, d\bar{z} \right) dz, \tag{6}$$

where z and $\bar{z}$ are formally treated as independent variables.

Hence, a particular solution of (1) is

$$u(x, y) = \operatorname{Re} \int \left(\int H(z, \bar{z})\, d\bar{z} \right) dz. \tag{7}$$

Remark. The function $G(x. y)$ in (4) is arbitrarly chosen. It can be taken to be zero. In that case (7) becomes

$$u(x, y) = \frac{1}{2} \operatorname{Re} \int \left(\int F\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i} \right) d\bar{z} \right) dz. \tag{8}$$

Example 1. Consider the Poisson equation

$$u_{xx} + u_{yy} = -xy.$$

If we apply (8) we find

$$u(x, y) = -\frac{1}{12} xy (x^2 + y^2).$$

On the other hand, if we take $G(x, y) = \frac{1}{8}(x^2 - y^2)$, from (7) we get

$$u(x, y) = \operatorname{Re} \frac{i}{24}(x^2 + y^2)(x^2 + 2ixy - y^2) = -\frac{1}{12} xy (x^2 + y^2).$$

3. In this section we consider two Dirichlet type problems for Poisson's equation (1). It will be more convenient to use polar coordinates (r, θ), defined by $x = r \cos \theta$, $y = r \sin \theta$. Then equation (1) becomes

$$u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = \Phi(r, \theta) \tag{9}$$

where $\Phi(r, \theta) = F(r \cos \theta, r \sin \theta)$.

Let $f: \mathbf{C} \to \mathbf{C}$ be a 2π-periodic function with the property that $f(t) \in \mathbf{R}$ for $t \in \mathbf{R}$. We consider the following problems:

Problem (P_1). *Find the function* $u: \mathbf{R}^2 \to \mathbf{R}$ *such that*

(i) u *satisfies* (9) *and is continuous in the region* $D = \{(r, \theta) \mid 0 \leqslant r \leqslant R\}$,

(ii) $u(R, \theta) = f(\theta)$;

Problem (P_2). *Find the function* $u: \mathbf{R}^2 \to \mathbf{R}$ *such that*

(i) u *satisfies* (9) *and is continuous in the region* $D_0 = \{(r, \theta) \mid 0 < r \leqslant R\}$,

(ii) $u(R, \theta) = f(\theta)$.

Remark. The only difference between problems (P_1) and (P_2) is that (P_2) in the point $r = 0$ is excluded. We note, in passing, that (P_2) seems more "natural" than (P_1) because the starting equation (9) is not defined for $r = 0$.

Suppose that $p(r, \theta)$ is a particular solution of equation (1). Then every function u defined by

$$(10) \qquad u(r, \theta) = p(r, \theta) + \mathrm{Re}\left(f\left(\theta + i\log\frac{r}{R}\right) - p\left(R, \theta + i\log\frac{r}{R}\right) + g(re^{i\theta}) - g\left(\frac{R^2}{r}e^{i\theta}\right)\right)$$

where g is an "arbitrary" function, is a solution of (P_2).

Problem (P_2) therefore has an infinity of solutions, given by (10). However, examples indicate that formula (10) also contains the unique solution of (P_1). In connection with this, see the next section.

Example 2. Let $\Phi(r, \theta) = -\frac{1}{2} r^2 \sin 2\theta$, $R = 1$, and $f(\theta) = \sin\theta$.

A particular solution of equation

$$(11) \qquad u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = -\frac{1}{2} r^2 \sin 2\theta$$

is $p(r, \theta) = -\frac{1}{24} r^4 \sin 2\theta$ (see Example 1).

Hence, the solutions of (P_2) for (11) are given by

$$(12) \qquad u(r, \theta) = -\frac{1}{24} r^4 \sin 2\theta + \frac{1}{2}\left(r + \frac{1}{r}\right)\sin\theta + \frac{1}{48}\left(r^2 + \frac{1}{r^2}\right)\sin 2\theta + \mathrm{Re}\left(g(re^{i\theta}) - g\left(\frac{1}{r}e^{i\theta}\right)\right).$$

If we want (12) to be continuous in $D = \{(r, \theta) \mid 0 \leqslant r \leqslant 1\}$, we have to destroy the term $\frac{1}{2r}\sin\theta + \frac{1}{48 r^2}\sin 2\theta$ in (12). We therefore choose $g(t) = -\frac{1}{2} it - \frac{1}{48} it^2$. For that choice of g, formula (12) yields the unique solution of (P_1), namely

$$u(r, \theta) = \frac{1}{24} r^2 (1 - r^2) \sin 2\theta + r\sin\theta.$$

4. We shall now develop a different formula which contains an infinity of solutions of (P_2).

Let $U(r, \theta)$ be the unique solution of (P_1). Then, if we put

$$u(r, \theta) = U(r, \theta) + V(r, \theta)$$

we see that $V(r, \theta)$ has to satisfy the Laplace equation and the boundary condition $V(R, \theta) = 0$. Hence (see [1]), we have

$$V(r, \theta) = \mathrm{Re}\left(h(re^{i\theta}) - h\left(\frac{R^2}{r}e^{i\theta}\right)\right),$$

and therefore

$$(13) \qquad u(r, \theta) = U(r, \theta) + \mathrm{Re}\left(h(re^{i\theta}) - h\left(\frac{R^2}{r}e^{i\theta}\right)\right),$$

where h is an "arbitrary" function.

Both formulas (10) and (13) contain an infinity of solutions to (P_2). It would be of interest to find whether (10) and (13) yield the same solutions, or whether one of them contains solutions which are not contained by the other.

Example 3. If we again consider the equation (11) with the boundary condition $u(1, \theta)=\sin\theta$, formula (10) reads

$$u(r, \theta)=-\frac{1}{24} r^4 \sin 2\theta+\frac{1}{2}\left(r+\frac{1}{r}\right)\sin\theta+\frac{1}{48}\left(r^2+\frac{1}{r^2}\right)\sin 2\theta + \operatorname{Re}\left(g(re^{i\theta})-g\left(\frac{1}{r}e^{i\theta}\right)\right) \tag{14}$$

and formula (13) becomes

$$u(r, \theta)=\frac{1}{24} r^2(1-r^2)\sin 2\theta+r\sin\theta+\operatorname{Re}\left(h(re^{i\theta})-h\left(\frac{1}{r}e^{i\theta}\right)\right). \tag{15}$$

In this case, letting $g(t)=h(t)-\frac{1}{2}it-\frac{1}{48}it^2$, we see that (14) and (15) coincide.

REFERENCE

[1] J. D. Kečkić, *A note on an incorrectly posed problem for the Laplace equation.* Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. № **412** — № **460** (1973), 83—84.