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A NEW CHARACTERIZATION OF COMPACT SETS IN FUNCTION SPACES

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Introduction

Gale [1] and Kelley and Morse [3] have given two different characterizations of compact sets in function spaces. The main purpose of this paper is to present a new such characterization (Theorems 2.7 and 3.3). Furthermore, Proposition 1.3 and Theorem 2.5 characterize evenly continuous families. In section 4, from Theorem 3.3, we derive Gale's theorem.

Notation, basic definitions and some known results. If X and Y are two arbitrary topological spaces, let Y^X denote the space of all continuous functions from X into Y with the compact-open topology, which, we recall, has as a subbasis the totality of all sets $(K, U)=\{f\in Y^X \mid f(K)\subset U\}$, where $K\subset X$ is compact and $U\subset Y$ is open.

Further, for any given point $a\in X$, let $p_a : Y^X\to Y$ denote the continuous [2, p. 165] function (called the *projection* determined by a) defined by $p_a(f)= =f(a)$ for every $f\in Y^X$.

Also, we say that a family $\mathbf{F}\subset Y^X$ is *evenly continuous* [3, p. 309] iff for each x in X, each y in Y, and each open neighborhood V of y there exists an open neighborhood U of x and an open neighborhood W of y such that if $f\in\mathbf{F}$ and $f(x)\in W$ then $f(U)\subset V$.

A space X is said to be a *k-space* [5, p. 285] iff the following condition holds: $A\subset X$ is open iff $A\cap C$ is open in C for each compact $C\subset X$.

We will use the following two remarkable theorems in the proof of both of our basic theorems ($\bar{A}$ denotes the closure of A):

Theorem K. (Kelley and Morse [3]) *Let X be a locally compact regular space, Y a regular Hausdorff space and $\mathbf{F}$ a subset of Y^X. Then $\mathbf{F}$ is compact if and only if the following conditions are satisfied:*

(A1) $\mathbf{F}$ *is closed in* Y^X,

(A2) $\overline{p_x(\mathbf{F})}$ *is compact in* Y *for each* $x\in X$,

(K3) $\mathbf{F}$ *is evenly continuous.*

Theorem M. (Kelley and Morse [3]) *Let X be a Hausdorff k-space, Y a regular Hausdorff space and* $\mathbf{F}$ *a subset of* Y^X. *Then* $\mathbf{F}$ *is compact if and only if the conditions* (A1), (A2) *and*

(M3) $\mathbf{F}|C=\{f|C|f\in\mathbf{F}\}$ *is an evenly continuous family in* Y^C *for every compact set C in X,*
are satisfied.

Further, throughout this paper, $P(Y)$ denotes the partitive set of Y taken with the Vietoris topology in which subbasic open sets are of the form

$$\langle U\rangle=\{S\in P(Y)\,|\,S\subset U\} \quad \text{and} \quad \rangle U\langle=\{S\in P(Y)\,|\,S\cap U\neq\varnothing\},$$

where U is an open set in Y. By 2^Y we denote the subspace of $P(Y)$ consisting of all closed subsets of Y.

Finally, we recall that, by definition, a function F from X into $P(Y)$ (or 2^Y) is *upper semi-continuous* (u.s.c.) if $F^{-1}(\rangle B\langle)$ is closed in X whenever B is closed in Y, or equivalently [4, vol. I, p. 185], if whenever $x\in X$, $V\subset Y$ is open and $F(x)\subset V$, there exists in X an open set $U\ni x$ such that $a\in U$ implies $F(a)\subset V$.

1. Some preliminary propositions

In this first section a simple characterization of evenly continuous family of functions and a remark on Theorems K and M are given.

Lemma 1.1. *Let* $\mathbf{F}$ *be an evenly continuous family in* Y^X *and C an arbitrary subset of X. Then* $\mathbf{F}|C=\{f|C|f\in\mathbf{F}\}$ *is also an evenly continuous family in* Y^C.

Proof. Let $x\in C$, $y\in Y$ and an open set $V\ni y$ be given. Since $\mathbf{F}$ is evenly continuous, there are in X and Y, respectively, open sets $G\ni x$ and $H\ni y$ such that

$$f\in\mathbf{F}\wedge f(x)\in H\Rightarrow f(G)\subset V. \tag{1}$$

Put
$$U=G\cap C, \qquad W=H.$$

Then U is an open neighborhood of x in C. Let

$$g\in\mathbf{F}|C, \quad \text{and} \quad g(x)\in W.$$

Then there exists $f\in\mathbf{F}$ such that $f|C=g$. From $x\in C$ and $f|C=g$ it follows that $f(x)=g(x)$ and, therefore, $f(x)\in H$. Consequently, by (1), $f(G)\subset V$ and so $f(U)\subset V$. Since $U\subset C$ and $f|C=g$, we have, finally, $g(U)\subset V$.

Thus, $\mathbf{F}|C$ is an evenly continuous family in Y^C.

Lemma 1.2. *Let X be a locally compact space and* $\mathbf{F}\subset Y^X$. *If* $\mathbf{F}|C$ *is an evenly continuous family in* Y^C *for every compact set C in X, then* $\mathbf{F}$ *is an evenly continuous family in* Y^X.

Proof. Let $x\in X$, $y\in Y$ and an open set $V\ni y$ be given. Since X is locally compact, there exists a compact set C such that $x\in\operatorname{int}(C)$. Since $\mathbf{F}|C$ is evenly continuous, there are in C and Y, respectively, open sets $N=G\cap C\ni x$ (G open in X) and $H\ni y$ such that

$$g\in\mathbf{F}|C\wedge g(x)\in H\Rightarrow g(N)\subset V. \tag{2}$$

Put

$$U = G \cap \operatorname{int}(C), \qquad W = H.$$

Obviously, $U \subset G \cap C = N$. Let $f \in \mathbf{F}$ and $f(x) \in W$. Put $g = f \mid C$. Then $g \in \mathbf{F} \mid C$, $f(x) = g(x)$ and hence $g(x) \in H$. Thus, by (2), $g(N) \subset V$ and, therefore, $g(U) \subset V$. Since $U \subset C$ and $g = f \mid C$, we have, finally, $f(U) \subset V$.

Lemmas 1.1 and 1.2 imply

Proposition 1.3. *Let X be a locally compact space and $\mathbf{F} \subset Y^X$. Then $\mathbf{F}$ is an evenly continuous family in Y^X if and only if $\mathbf{F} \mid C$ is an evenly continuous family in Y^C for every compact set C in X.*

Remark 1.4. It is known that every locally compact space is a k-space [5, p. 285]. But, obviously, a locally compact regular space need not be a Hausdorff space. Thus, in general case, Theorems K and M are independent. If, however, X is a locally compact Hausdorff space then, by Proposition 1.3, Theorem K follows from Theorem M.

2. A function related to even continuity

Let X and Y be arbitrary topological spaces and let an arbitrary subset $\mathbf{F}$ of Y^X be given. Then every subset $\mathbf{\Phi}$ of $\mathbf{F}$ induces a natural function

$$\overset{*}{F}_{\Phi} : X \to 2^Y$$

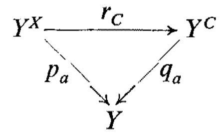
defined by

$$\overset{*}{F}_{\Phi}(x) = \overline{p_x(\mathbf{\Phi})}$$

for each $x \in X$ (i. e., $\overset{*}{F}_{\Phi}(x) = \overline{\{f(x) \mid f \in \mathbf{\Phi}\}}$). This function plays a very important role in all ours considerations.

Lemma 2.1. *Let C be an arbitrary subset of a space X and a any point in C. Then*

(a) *The diagram**



commutes,

(b) *Functions $\overset{*}{F}_{\Phi \mid C}$ and $\overset{*}{F}_{\Phi} \mid C$ are equal.*

Proof. (a) For $f \in Y^X$, we have

$$(q_a \circ r_C)(f) = q_a(f \mid C) = (f \mid C)(a) = f(a) = p_a(f),$$

and so $p_a = q_a \circ r_C$.

(b) Let $x \in C$ be arbitrarily given. Then, by (a),

$$\overset{*}{F}_{\Phi \mid C}(x) = \overline{q_x(\mathbf{\Phi} \mid C)} = \overline{q_x[r_C(\mathbf{\Phi})]} = \overline{p_x(\mathbf{\Phi})} = \overset{*}{F}_{\Phi}(x) = (\overset{*}{F}_{\Phi} \mid C)(x).$$

* Here p_a, q_a denote projections and $r_C : Y^X \to Y^C$ is defined by $r_C(f) = f \mid C$ for every $f \in Y^X$.

Lemma 2.2. *Let X be an arbitrary and Y a regular topological space and let $\mathbf{F} \subset Y^X$. If $\overset{*}{F}_\Phi$ is upper semi-continuous for every closed subset Φ of $\mathbf{F}$, then $\mathbf{F}$ is an evenly continuous family.*

Proof. Let $x \in X$, $y \in Y$ and an open set $V \ni y$ be arbitrarily given. Since Y is a regular space, there exists in Y an open set G such that

$$y \in G, \qquad \bar{G} \subset U.$$

Thus,

$$\Phi = (x, \bar{G}) \cap \mathbf{F}$$

is closed in $\mathbf{F}$ [4, vol. II, p. 85, Theorem 1]. Obviously, $p_x(\Phi) \subset \bar{G}$. This implies $\overset{*}{F}_\Phi(x) \subset \bar{G}$ and hence, clearly, $\overset{*}{F}_\Phi(x) \subset V$. Therefore, since $\overset{*}{F}_\Phi$ is u.s.c., there exists in X an open set $U \ni x$ such that

$$a \in U \Rightarrow \overset{*}{F}_\Phi(a) \subset V.$$

Put $W = G$ and let us show that

$$\mathbf{F} \cap (x, W) \subset (U, V).$$

Let $f \in \mathbf{F} \cap (x, W)$ and $a \in U$. Then $f \in \Phi$ and, thus, $p_a(f) \in p_a(\Phi)$. Hence $f(a) \in \overline{p_a(\Phi)} = \overset{*}{F}_\Phi(a)$. Since $\overset{*}{F}_\Phi(a) \subset V$, we have $f(a) \in V$ and, therefore, $f(U) \subset V$ i. e., $f \in (U, V)$. This shows that $\mathbf{F}$ is evenly continuous.

As an immediate consequence of 2.2 and [5, p. 286, Theorem 43.14] we have the following proposition:

Proposition 2.3. *Let X be an arbitrary and Y a regular topological space and let $\mathbf{F} \subset Y^X$. If $\overset{*}{F}_\Phi$ is* u s.c. *for every closed subset Φ of $\mathbf{F}$, then, on $\mathbf{F}$, the compact-open topology reduces to the point-open topology.*

The following lemma was suggested to me by M. M. Marjanović.

Lemma 2.4. *Let X be an arbitrary and Y a compact Hausdorff space. If $\mathbf{F} \subset Y^X$ is an evenly continuous family, then $\overset{*}{F}_\Phi$ is* u. s. c. *for every subset Φ of $\mathbf{F}$.*

Proof. Let $\Phi \subset \mathbf{F}$, $x \in X$ and a set V open in Y such that $\overset{*}{F}_\Phi(x) \subset V$ be given. Since Y is regular and $\overset{*}{F}_\Phi(x)$ compact there exists in Y an open set G such that

(4) $$\overset{*}{F}_\Phi(x) \subset G, \qquad \bar{G} \subset V.$$

Choose arbitrarily a point $y \in F_\Phi(x) \subset G$. Clearly, since $\mathbf{F}$ is evenly continuous, so also is Φ and, therefore, there are in X and Y, respectively, open sets $U_y \ni x$ and $W_y \ni y$ such that

(5) $$f \in \Phi \wedge f(x) \in W_y \Rightarrow f(U_y) \subset G.$$

Thus, $\{W_y \mid y \in \overset{*}{F}_\Phi(x)\}$ is an open cover of the compact set $\overset{*}{F}_\Phi(x)$ and, hence, there is a finite number of points $y_1, \ldots, y_n \in \overset{*}{F}_\Phi(x)$ such that

$$\overset{*}{F}_\Phi(x) \subset W_{y_1} \cup \cdots \cup W_{y_n}.$$

Put

$$U = U_{y_1} \cap \cdots \cap U_{y_n}.$$

Since U is open and contains x, it remains to prove that

$$a \in U \Rightarrow \overset{*}{F}_\Phi(a) \subset V.$$

Indeed, if $f \in \Phi$, then $f(x) \in \overset{*}{F}_\Phi(x)$ and so $f(x) \in W_{y_i}$ for some $i \leqslant n$. Thus, by (5), $f(U_{y_i}) \subset G$ and, consequently, $f(U) \subset G$ i. e., $f(a) \in G$. Therefore,

$$\{f(a) \mid f \in \Phi\} \subset G.$$

Hence $\overline{p_a(\Phi)} \subset \overline{G}$, and, by (4), $\overset{*}{F}_\Phi(a) \subset V$.

From Lemmas 2.2 and 2.4 we infer the following theorem:

Theorem 2.5. *Let Y be a compact Hausdorff space and $\mathbf{F} \subset Y^X$. Then the following are equivalent:*

(a) $\mathbf{F}$ *is evenly continuous,*

(b) $\overset{*}{F}_\Phi$ *is* u. s. c. *for every closed subset* Φ *of* $\mathbf{F}$.

Lemma 2.6. *Let X be a locally compact regular space. Y a Hausdorff space and $\mathbf{F}$ a compact set in Y^X. Then $\overset{*}{F}_\Phi$ is* u. s. c. *for every closed subset Φ of $\mathbf{F}$.*

Proof. Let a closed subset Φ of $\mathbf{F}$, $x \in X$ and an open set V in Y such that $\overset{*}{F}_\Phi(x) \subset V$ be given. Let us find an open neighborhood U of the point x such that

$$a \in U \Rightarrow \overset{*}{F}_\Phi(a) \subset V. \tag{6}$$

First, let us observe that $p_a(\Phi)$ is closed is Y (for each $a \in X$). Indeed, since Φ is closed in $\mathbf{F}$ and $\mathbf{F}$ is compact, it follows that Φ is a compact set and, thus (p_a is continuous), so is $p_a(\Phi)$. But, because Y is a Hausdorff space, we have $\overline{p_a(\Phi)} = p_a(\Phi)$, i. e., $\overset{*}{F}_\Phi(a) = \{f(a) \mid f \in \Phi\}$. Hence, by (6), we must show that

$$a \in U \wedge f \in \Phi \Rightarrow f(a) \in V. \tag{7}$$

Let $g \in \Phi$ be arbitrarily given. Then $g(x) \in \overset{*}{F}_\Phi(x)$, and so $g(x) \in V$. Since g is continuous, there exists in X an open set $U_g \ni x$ such that

$$g(U_g) \subset V.$$

Since X is a locally compact regular space, there exists [2, p. 66. Proposition 2.14] a compact set K_g such that

$$x \in \operatorname{int}(K_g), \qquad K_g \subset U_g.$$

Thus, $g(K_g) \subset V$, or, $g \in (K_g, V)$. Therefore, the family $\{(K_g, V) \mid g \in \Phi\}$ is an open cover of the compact set Φ and, hence, there is a finite number of points $g_1, \ldots, g_n \in \Phi$ such that

$$\Phi \subset (K_{g_1}, V) \cup \cdots \cup (K_{g_n}, V).$$

Let

$$U = \operatorname{int}(K_{g_1} \cap \cdots \cap K_{g_n}) = \operatorname{int}(K_{g_1}) \cap \cdots \cap \operatorname{int}(K_{g_n}).$$

Since U is open and contains x, it remains to verify the implication (7). Let $f \in \Phi$ and $a \in U$. Then $f(K_{g_i}) \subset V$ for some $i \leqslant n$ and $a \in \operatorname{int}(K_{g_i})$. This implies $f(a) \in V$, which completes the proof.

Now, we prove the first of two theorems characterizing compact sets in Y^X.

Theorem 2.7. *Under the same hypothesis for X and Y as in Theorem* K, $\mathbf{F}\subset Y^X$ *is compact if and only if the conditions* (A1), (A2), *and*

(C3) $\overset{*}{F}_\Phi$ *is* u.s.c. *for every closed subset* Φ *of* $\mathbf{F}$,

are satisfied.

Proof. Obviously, by Theorem K, it is sufficient to show that

$$(A1)\wedge(A2)\wedge(C3)\Leftrightarrow(A1)\wedge(A2)\wedge(K3).$$

$\Rightarrow$: By Lemma 2.2.

$\Leftarrow$: $\mathbf{F}$ is compact (Theorem K); the implication follows according to Lemma 2.6.

From Theorems 2.7. and K we have (compare with 2.5):

Corollary 2.8. *Let X be a locally compact regular space, Y a regular Hausdorff space and* $\mathbf{F}$ *a subset of Y^X satisfying* (A1) *and* (A2). *Then* (K3) *and* (C3) *are equivalent.*

3. Proof of the main theorem

Let us prove first two lemmas.

Lemma 3.1. *Let X and Y be Hausdorff spaces and* $\mathbf{F}$ *a compact set in Y^X. If C is a compact set in X and Φ is a closed set in* $\mathbf{F}$, *then the restriction* $\overset{*}{F}_\Phi|C$ *is* u.s.c..

Proof. Obviously, C is compact Hausdorff space and, thus, a locally compact regular space.

Since $r_C: Y^X\to Y^C$ is continuous [4, vol. II, p. 91, Theorem 1] and $\mathbf{F}$ is a compact set, $r_C(\mathbf{F})=\mathbf{F}|C$ is a compact set in the space Y^C.

Now, let us show that $\Phi|C$ is a closed set in $\mathbf{F}|C$. Indeed, since Φ is closed in $\mathbf{F}$ and $\mathbf{F}$ is compact, Φ is compact in Y^X and, hence, $r_C(\Phi)=\Phi|C$ is compact in Y^C. Since Y is a Hausdorff space, so is Y^C [2, p. 151, Proposition 1.1] and, thus, $\Phi|C$ is closed in Y^C. This together with $\Phi|C\subset\mathbf{F}|C$ implies that $\Phi|C$ is closed in $\mathbf{F}|C$.

Therefore, according to Lemma 2.6. (where X is to be replaced by C, Y^X by Y^C, $\mathbf{F}$ by $\mathbf{F}|C$ and Φ by $\Phi|C$), $\overset{*}{F}_{\Phi|C}$ is u.s.c.. But, by Lemma 2.1, $\overset{*}{F}_{\Phi|C}=\overset{*}{F}_\Phi|C$ and, thus, $\overset{*}{F}_\Phi|C$ is u.s.c..

Lemma 3.2. *Let X be a k-space and $F: X\to 2^Y$ an arbitrary map. If the restriction $F|C$ is* u.s.c. *for every compact set C in X, then F is also* u.s.c..

Proof. Let U be an open set in Y. Let us prove that $F^{-1}(\langle U\rangle)$ is open in X. Let C be an arbitrary compact set in X. Obviously, we have

$$F^{-1}(\langle U\rangle)\cap C=(F|C)^{-1}(\langle U\rangle).$$

Since $F|C$ is u.s.c., $(F|C)^{-1}(\langle U\rangle)$ is an open set in C. Consequently, since $F^{-1}(\langle U\rangle)\cap C$ is open in C and is a k-space, $F^{-1}(\langle U\rangle)$ is open in X.

We now state the main result.

Theorem 3.3. *Under the same hypothesis for X and Y as in Theorem* M, $\mathbf{F} \subset Y^X$ *is compact if and only if the conditions* (A1), (A2) *and* (C3) *are satisfied.*

Proof. By Theorem M, it is sufficient to show that

$$(A1) \wedge (A2) \wedge (C3) \Leftrightarrow (A1) \wedge (A2) \wedge (M3).$$

$\Rightarrow$: By Lemma 2.2, $\mathbf{F}$ is an evenly continuous family, and, by Lemma 1.1, $\mathbf{F}|C$ is also an evenly continuous family for every compact set C in X.

$\Leftarrow$: By Theorem M, $\mathbf{F}$ is compact, and hence, by Lemma 3.1, $\overset{*}{F}_\Phi|C$ is u.s.c. for every compact set C in X and every closed subset Φ of $\mathbf{F}$. From this, we conclude (Lemma 3.2) that $\overset{*}{F}_\Phi$ is u.s.c. for every closed subset Φ of $\mathbf{F}$.

Theorems 3.3 and M imply

Corollary 3.4. *Let X be a Hausdorff k-space, Y a regular Hausdorff space and* $\mathbf{F}$ *a subset of Y^X satisfying* (A1) *and* (A2). *Then* (M3) *and* ($C3$) *are equivalent.*

Remark 3.5. Let us observe that the condition (C3), by 2.1 and 3.2, can be replaced by the following condition:

(C4) $\overset{*}{F}_{\Phi|C}$ *is* u.s.c. *for every compact set C in X and every closed subset* Φ *of* $\mathbf{F}$.

4. On Gale's theorem

In this section, from Theorem 3.3, we infer Gale's theorem (see 4.5).

Let X and Y be two arbitrary topological spaces and let $\mathbf{F} \subset Y^X$ be given. Then every subset Φ of $\mathbf{F}$ induces a function

$$F_\Phi : X \to P(Y)$$

defined by

$$F_\Phi(x) = p_x(\Phi)$$

for each $x \in X$. Although of technical nature, the following lemma is very important as the basis of the connection between the above mentioned theorems.

Lemma 4.1. *Let B be an arbitrary subset of Y. Then*

$$F_\Phi^{-1}(\rangle B \langle) = \cup\{f^{-1}(B) \mid f \in \Phi\}.$$

Proof. This follows from

$$x \in F_\Phi^{-1}(\rangle B\langle) \Leftrightarrow F_\Phi(x) \in \rangle B\langle \Leftrightarrow \{f(x) \mid f \in \Phi\} \cap B \neq \varnothing$$
$$\Leftrightarrow (\exists f)(f \in \Phi \wedge f(x) \in B) \Leftrightarrow (\exists f)(f \in \Phi \wedge x \in f^{-1}(B)).$$

Lemma 4.2. *The following are equivalent:*

(G3) *If B is closed in Y and Φ is closed in* $\mathbf{F}$, *then* $\cup\{f^{-1}(B) \mid f \in \Phi\}$ *is closed in X,*

(H3) F_Φ *is* u.s.c. *for every closed subset* Φ *of* $\mathbf{F}$.

Proof. This assertion follows easily from Lemma 4.1.

According to Lemma 4.2, Gale's theorem obtains the form very similar to the form of Theorem 3.3. This, however, makes it possible for us to achieve the desirable aim quite easily by using the next two simple lemmas.

Lemma 4.3. *Let Y be a Hausdorff space,* $\mathbf{F}$ *a compact set in* Y^X *and* $\mathbf{\Phi}$ *a closed subset of* $\mathbf{F}$. *Then functions* F_Φ *and* $\overset{*}{F}_\Phi$ *are equal.*

Proof. By definition of functions F_Φ and $\overset{*}{F}_\Phi$, it is sufficient to show that $p_x(\mathbf{\Phi})=\overline{p_x(\mathbf{\Phi})}$ for each $x\in X$. The proof of this equality is, however, contained in the proof od Lemma 2.6.

Lemma 4.4. *Let Y be a regular space and* $\mathbf{F}$ *a subset of* Y^X. *Then conditions*

(G2) $p_x(\mathbf{F})$ *is compact in Y for each $x\in X$*

and (G3) *imply conditions* (A2) *and* (C3).

Proof. (A2) follows from (G2) and regularity of Y[2, p. 70, exercise $2F$]. Let us prove (C3).

Let a closed subset $\mathbf{\Phi}$ of $\mathbf{F}$, $x\in X$ and a set V open in Y such that $\overset{*}{F}_\Phi(x)=\overline{p_x(\mathbf{\Phi})}\subset V$ be given. As a closed subset of a compact space $\overline{p_x(\mathbf{F})}$, $\overline{p_x(\mathbf{\Phi})}$ is also compact. Therefore, since Y is a regular space, there exists in Y an open set G such that

$$\overline{p_x(\mathbf{\Phi})}\subset G, \qquad \overline{G}\subset V.$$

Hence $F_\Phi(x)=p_x(\mathbf{\Phi})\subset G$, and, since (Lemma 4.2.) F_Φ is u.s.c., there exists in X an open set $U\ni x$ such that

$$a\in U \Rightarrow F_\Phi(a)=p_a(\mathbf{\Phi})\subset G.$$

Hence

$$\overset{*}{F}_\Phi(a)=\overline{p_a(\mathbf{\Phi})}\subset\overline{G}\subset V,$$

which completes the proof.

From Theorem 3.3, using 4.3 and 4.4, we now derive immediatelly.

Theorem 4.5. (Gale [1]) *Under the same hypothesis for X and Y as in Theorem* M, $\mathbf{F}\subset Y^X$ *is compact if and only if the following conditions are satisfied:*

(A1) $\mathbf{F}$ *is closed in* Y^X,

(G2) $p_x(\mathbf{F})$ *is compact in Y for each $x\in X$,*

(G3) *If B is closed in Y and* $\mathbf{\Phi}$ *is closed in* $\mathbf{F}$, *then* $\bigcup\{f^{-1}(B)\,|\,f\in\mathbf{\Phi}\}$ *is closed in X.*

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