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# SOME BILINEAR AND BILATERAL RELATIONS FOR HYPERGEOMETRIC FUNCTIONS OF THREE VARIABLES

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**1.** Srivastava (see references [1], [2], and [3]) has obtained bilinear and bilateral generating relations for hypergeometric functions of two variables. In this paper, we have made an attempt to extend these results for hypergeometric functions of three variables, viz. $F^{(3)}[x, y, z]$ defined by Srivastava [4, p. 428] in the form

$$F^{(3)}\left[\begin{matrix}(a)::(b);\ (b');\ (b''):(c);\ (c');\ (c'');\\ (e)::(f);\ (f');\ (f''):(g);\ (g');\ (g'');\end{matrix}\ x, y, z\right]$$

$$(1.1)\qquad =\sum_{m,n,p=0}^{\infty}\frac{((a))_{m+n+p}((b))_{m+n}((b'))_{n+p}((b''))_{m+p}((c))_m((c'))_n((c''))_p\, x^m y^n z^p}{((e))_{m+n+p}((f))_{m+n}((f'))_{n+p}((f''))_{m+p}((g))_m((g'))_n((g''))_p\, m!\, n!\, p!},$$

where $(a)$ denotes the sequence of $A$ parameters $a_1, \ldots, a_A$. It will be assumed throughout the paper that there are $A$ of the $a$ parameters, $B$ of the $b$ parameters, and so on. Thus $((a))_m$ is to be interpreted as $\prod_{j=1}^{A}(a_j)_m$, with similar interpretations for $((b))_m$ etc. The Srivastava function $F^{(3)}[x, y, z]$ provides a unification of Lauricella's fourteen hypergeometric functions of three variables, viz. $F_1, \ldots, F_{14}$ (see [5], p. 114), and of three additional functions $H_A$, $H_B$, $H_C$ defined by Srivastava himself [6, p. 97] A fairly large number of special cases, known or new, can be obtained by appropriate choices of the parameters or the variables involved.

We first prove the following formulae:

$$F_G(a, a, a, b_1, b_2, b_3;\ c_1, c_2, c_2;\ x, y, z)$$

$$(1.2)\qquad =\frac{\Gamma(c_1)}{\Gamma(b_1)}x^{1-c_1}D_x^{b_1-c_1}\left[x^{b_1-1}(1-x)^{-a}F_1\left(a, b_2, b_3;\ c_2;\ \frac{y}{1-x}, \frac{z}{1-x}\right)\right]$$

and

$$F_K(a_1, a_2, a_2, b_1, b_2, b_1;\ c_1, c_2, c_3;\ x, y, z)$$

$$(1.3)\qquad =\frac{\Gamma(c_2)}{\Gamma(b_2)}y^{1-c_2}D_y^{b_2-c_2}\left[y^{b_2-1}(1-y)^{-a_2}F_2\left(b_1, a_1, a_2;\ c_1, c_3;\ x, \frac{z}{1-y}\right)\right],$$

where $D_x^{\lambda}$ is an operator of fractional derivative defined by

$$D_x^{\lambda}(x^{\mu-1})=\frac{\Gamma(\mu)}{\Gamma(\mu-\lambda)}x^{\mu-\lambda-1},$$

and $F_G$, $F_K$ are the same as the Lauricella functions $F_8$ and $F_3$ [5, p. 114] of three variables in the subsequent notation of Saran [7].

Proofs of (1.2) and (1.3): We express the functions $F_G$ and $F_K$ in series forms as

$$F_G(a, a, a, b_1, b_2, b_3; c_1, c_2, c_2; x, y, z)$$

$$(1.4)\qquad =\sum_{m,n,p=0}^{\infty}\frac{(a)_{m+n+p}(b_1)_m(b_2)_n(b_3)_p}{(c_1)_m(c_2)_{n+p}}\frac{x^m y^n z^p}{m!\,n!\,p!}$$

and

$$F_K(a_1, a_2, a_2, b_1, b_2, b_1; c_1, c_2, c_3; x, y, z)$$

$$(1.5)\qquad =\sum_{m,n,p=0}^{\infty}\frac{(a_1)_m(a_2)_{n+p}(b_1)_{m+p}(b_2)_n}{(c_1)_m(c_2)_n(c_3)_p}\frac{x^m y^n z^p}{m!\,n!\,p!}.$$

Now making use of the results

$$D_x^{b_1-c_1}(x^{b_1+m-1})=\frac{\Gamma(b_1+m)}{\Gamma(c_1+m)}x^{c_1+m-1}$$

and

$$D_y^{b_2-c_2}(y^{b_2+n-1})=\frac{\Gamma(b_2+n)}{\Gamma(c_2+n)}y^{c_2+n-1},$$

in (1.4) and (1.5), respectively, we obtain on simplification the results (1.2) and (1.3).

**2.** In this section, we obtain the following bilateral generating relations involving Lauricella functions of three variables in the subsequent notation of Saran [7].

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}F_G(a, a, a, c', -n, c''; h', g'', g''; y, x, z)$$

$$\cdot F_K(-n, v', v', v'', w', v''; l, l', l''; \xi, \eta, \rho)\,t^n$$

$$=(1-t)^{-\lambda}\sum_{n=0}^{\infty}\frac{(\lambda)_n(a)_n(v'')_n}{n!\,(l)_n(g'')_n}\left[\frac{x\xi t}{(1-t)^2}\right]^n$$

$$\cdot F_G\left(a+n, a+n, a+n, c', \lambda+n, c''; h', g''+n, g''+n; y, \frac{xt}{t-1}, z\right)$$

$$(2.1)\qquad \cdot F_K\left(\lambda+n, v', v', v''+n, w', v''+n; l+n, l', l''; \frac{\xi t}{t-1}, \eta, \rho\right)$$

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} F_G(a, a, a, c', -n, c''; h', g'', g''; y, x, z)$$

$$\cdot F_K(\lambda+n, v', v', v'' w', v'', l, l', l''; \xi, \eta, \rho) t^n$$

$$= (1-t)^{-\lambda} \sum_{n=0}^{\infty} \frac{(\lambda)_n (a)_n (v'')_n}{n! \ (l)_n (g'')_n} \left[\frac{-x \xi t}{(1-t)^2}\right]^n$$

$$\cdot F_G\left(a+n, a+n, a+n, c', \lambda+n, c''; h', g''+n, g''+n; y, \frac{xt}{t-1}, z\right)$$

$$\cdot F_K\left(\lambda+n, v', v', v''+n, w', v''+n; l+n, l', l''; \frac{\xi}{1-t}, \eta, \rho\right) \tag{2.2}$$

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} F_G(a, a, a, c', \lambda+n, c''; h', g'', g''; y, x, z)$$

$$\cdot F_K(-n, v', v', v'', w', v''; l, l', l''; \xi, \eta, \rho) t^n$$

$$= (1-t)^{-\lambda} \sum_{n=0}^{\infty} \frac{(\lambda)_n (a)_n (v'')_n}{n! \ (l)_n (g'')_n} \left[\frac{-x \xi t}{(1-t)^2}\right]^n$$

$$\cdot F_G\left(a+n, a+n, a+n, c', \lambda+n, c''; h', g''+n, g''+n; y, \frac{x}{1-t}, z\right)$$

$$\cdot F_K\left(\lambda+n, v', v', v''+n, w', v''+n; l+n, l', l''; \frac{\xi t}{t-1}, \eta, \rho\right) \tag{2.3}$$

and

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} F_G(a, a, a, c', \lambda+n, c''; h', g'', g''; y, x, z)$$

$$\cdot F_K(\lambda+n, v', v', v'', w', v''; l, l', l''; \xi, \eta, \rho) t^n$$

$$= (1-t)^{-\lambda} \sum_{n=0}^{\infty} \frac{(\lambda)_n (a)_n (v'')_n}{n! \ (l)_n (g'')_n} \left[\frac{x \xi t}{(1-t)^2}\right]^n$$

$$\cdot F_G\left(a+n, a+n, a+n, c', \lambda+n, c''; h', g''+n, g''+n; y, \frac{x}{1-t}, z\right)$$

$$\cdot F_K\left(\lambda+n, v', v', v''+n, w', v''+n; l+n, l', l''; \frac{\xi}{1-t}, \eta, \rho\right). \tag{2.4}$$

Proof of (2.1): From [3, p. 83 (3.16)], we have

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} F_1(a, -n, c''; g''; x, z) F_2(v'', -n, v'; l, l''; \xi, \rho) t^n$$

$$= (1-t)^{-\lambda} \sum_{n=0}^{\infty} \frac{(\lambda)_n (a)_n (v'')_n}{n! \ (l)_n (g'')_n} \left[\frac{x \xi t}{(1-t)^2}\right]^n F_1\left(a+n, \lambda+n, c''; g''+n; \frac{xt}{t-1}, z\right)$$

$$\cdot F_2\left(v''+n, \lambda+n, v'; l+n, l''; \frac{\xi t}{t-1}, \rho\right).$$

Replacing $x$ by $\frac{x}{1-y}$, $z$ by $\frac{z}{1-y}$, $\rho$ by $\rho/(1-\eta)$, multiplying by $y^{c'-1}(1-y)^{-a}\eta^{w'-1}(1-\eta)^{-v'}$ and operating on both sides by $D_y^{c'-h'}$ and $D_\eta^{w'-l'}$, we get

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}\left[D_y^{c'-h'}\left\{y^{c'-1}(1-y)^{-a}F_1\left(a,\ -n,\ c'';\ g'';\ \frac{x}{1-y},\ \frac{z}{1-y}\right)\right\}\right]$$
$$\cdot\,[D_\eta^{w'-l'}\{\eta^{w'-1}(1-\eta)^{-v'}F_2(v'',\ -n,\ v';\ l,\ l'';\ \xi,\ \rho/(1-\eta))\}]\,t^n$$
$$=(1-t)^{-\lambda}\sum_{n=0}^{\infty}\frac{(\lambda)_n(a)_n(v'')_n}{n!\,(l)_n(g'')_n}\left[\frac{x\xi t}{(1-t)^2}\right]^n$$
$$\cdot\left[D_y^{c'-h'}\left\{y^{c'-1}(1-y)^{-a-n}F_1\left(a+n,\ \lambda+n,\ c'';\ g''+n;\ \frac{xt}{(t-1)(1-y)},\ \frac{z}{1-y}\right)\right\}\right]\cdot$$
$$\cdot\left[D_\eta^{w'-l'}\left\{\eta^{w'-1}(1-\eta)^{-v'}F_2\left(v''+n,\ \lambda+n,\ v';\ l+n,\ l'';\ \frac{\xi t}{t-1},\ \rho/(1-\eta)\right)\right\}\right].$$

Now employing the formulae (1.2) and (1.3) to the above, we arrive at (2.1). Similar are the proofs of (2.2) to (2.4). Use will be made of the known formulae [3, p. 83, (3.19), (3.20) and (3.25)].

3. *Extensions of the generating relations* (2.1) *to* (2.4).

Here we have made an attempt to extend the results of the last section to functions of three variables such as the $F^{(3)}[x, y, z]$ defined by (1.1). In obtaining these results, the principle of multidimensional mathematical induction as well as the Laplace and inverse Laplace transform techniques are used.

If for convenience, we denote the quotient

$$\frac{(\lambda)_n((a))_n((b))_n((b''))_n((c))_n((u))_n((v))_n((v''))_n((w))_n}{n!\,((e))_n((g))_n((g''))_n((h))_n((d))_n(k))_n((k''))_n((l))_n}$$

by

$$\delta\begin{bmatrix}\lambda,\ a,\ b,\ b'',\ c,\ u,\ v,\ v'',\ w\\ n,\ e,\ g,\ g'',\ h,\ d,\ k,\ k'',\ l\end{bmatrix},$$

then the relations to be estabilished here are

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}F^{(3)}\begin{bmatrix}(a)::(b);\ (b');\ (b''):\ -n,\ (c);\ (c');\ (c'');\\ (e)::(g);\ (g');\ (g''):\qquad (h);\ (h');\ (h'');\end{bmatrix}x,\ y,\ z\Bigg]$$
$$\cdot\,F^{(3)}\begin{bmatrix}(u)::(v);\ (v');\ (v''):\ -n,\ (w);\ (w');\ (w'');\\ (d)::(k);\ (k');\ (k''):\qquad (l);\ (l');\ (l'');\end{bmatrix}\xi,\ \eta,\ \rho\Bigg]t^n$$
$$=(1-t)^{-\lambda}\sum_{n=0}^{\infty}\delta\begin{bmatrix}\lambda,\ a,\ b,\ b'',\ c,\ u,\ v,\ v'',\ w\\ n,\ e,\ g,\ g'',\ h,\ d,\ k,\ k'',\ l\end{bmatrix}\left[\frac{x\xi t}{(1-t)^2}\right]^n$$
$$\cdot\,F^{(3)}\begin{bmatrix}(a)+n::(b)+n;\ (b');\ (b'')+n:\lambda+n,\ (c)+n;\ (c');\ (c'');\ \frac{xt}{t-1},\ y,\ z\\ (e)+n::(g)+n;\ (g');\ (g'')+n:\qquad (h)+n;\ (h');\ (h'');\end{bmatrix}$$
$$\cdot\,F^{(3)}\begin{bmatrix}(u)+n::(v)+n;\ (v');\ (v'')+n:\lambda+n,\ (w+n);\ (w');\ (w'');\ \frac{\xi t}{t-1},\ \eta,\ \rho\\ (d)+n::(k)+n;\ (k');\ (k'')+n:\qquad (l)+n;\ (l');\ (l'');\end{bmatrix},$$

(3.1)

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}F^{(3)}\left[\begin{matrix}(a)::(b);\ (b');\ (b''):-n,\ (c);\ (c');\ (c'');\\ (e)::(g);\ (g');\ (g''):\quad (h);\ (h');\ (h'');\end{matrix}\ x,y,z\right]$$
$$\cdot F^{(3)}\left[\begin{matrix}(u)::(v);\ (v');\ (v''):\lambda+n,\ (w);\ (w');\ (w'');\\ (d)::(k);\ (k');\ (k''):\quad (l);\ (l');\ (l'');\end{matrix}\ \xi,\eta,\rho\right]t^n$$
$$=(1-t)^{-\lambda}\sum_{n=0}^{\infty}\delta\left[\begin{matrix}\lambda,\ a,\ b,\ b'',\ c,\ u,\ v,\ v'',\ w\\ n,\ e,\ g,\ g'',\ h,\ d,\ k,\ k'',\ l\end{matrix}\right]\left[\frac{-x\xi t}{(1-t)^2}\right]^n$$
$$\cdot F^{(3)}\left[\begin{matrix}(a)+n::(b)+n;\ (b');\ (b'')+n:\lambda+n,\ (c)+n;\ (c');\ (c'');\\ (e)+n::(g)+n;\ (g');\ (g'')+n:\quad (h)+n;\ (h');\ (h'');\end{matrix}\ \frac{xt}{t-1},y,z\right]$$
$$\cdot F^{(3)}\left[\begin{matrix}(u)+n::(v)+n;\ (v');\ (v'')+n:\lambda+n,\ (w)+n;\ (w');\ (w'');\\ (d)+n::(k)+n;\ (k');\ (k'')+n:\quad (l)+n;\ (l');\ (l'');\end{matrix}\ \frac{\xi}{1-t},\eta,\rho\right],$$
(3.2)

and

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}F^{(3)}\left[\begin{matrix}(a)::(b);\ (b');\ (b''):\lambda+n;\ (c);\ (c');\ (c'');\\ (e)::(g);\ (g');\ (g''):\quad (h);\ (h');\ (h'');\end{matrix}\ x,y,z\right]$$
$$\cdot F^{(3)}\left[\begin{matrix}(u)::(v);\ (v');\ (v''):\lambda+n,\ (w);\ (w');\ (w'');\\ (d)::(k);\ (k');\ (k''):\quad (l);\ (l');\ (l'');\end{matrix}\ \xi,\eta,\rho\right]t^n$$
$$=(1-t)^{-\lambda}\sum_{n=0}^{\infty}\delta\left[\begin{matrix}\lambda,\ a,\ b,\ b'',\ c,\ u,\ v,\ v'',\ w\\ n,\ e,\ g,\ g'',\ h,\ d,\ k,\ k'',\ l\end{matrix}\right]\left[\frac{x\xi t}{(1-t)^2}\right]^n$$
$$\cdot F^{(3)}\left[\begin{matrix}(a)+n::(b)+n;\ (b');\ (b'')+n:\lambda+n,\ (c)+n;\ (c');\ (c'');\\ (e)+n::(g)+n;\ (g');\ (g'')+n:\quad (h)+n:\ (h');\ (h'');\end{matrix}\ \frac{x}{1-t},y,z\right]$$
$$\cdot F^{(3)}\left[\begin{matrix}(u)+n::(v)+n;\ (v');\ (v'')+n:\lambda+n,\ (w)+n;\ (w');\ (w'');\\ (d)+n::(k)+n;\ (k');\ (k'')+n:\quad (l)+n;\ (l');\ (l'');\end{matrix}\ \frac{\xi}{1-t},\eta,\rho\right].$$
(3.3)

Proof of (3.1): Clearly (3.1) is true for $A-1=B=B'=B''=C=C'-1=$ $=C''-1=E=G=G'=G''-1=H=H'-1=H''=U=V=V'-1=V''-1=W=$ $=W'-1=W''=D=K=K'=K''=L-1=L'-1=L''-1=0$, because of (2.1). For the proof of (3.1) by the method of multidimensional mathematical induction, let us assume it to be true for some values of the non-negative integers $A, B, B', \ldots, L''$. Replacing $x$ by $xt_1$, $y$ by $yt_1$, $z$ by $zt_1$ in (3.1), multiplying both sides by $(t_1)^{a_{A+1}-1}$ and taking their Laplace transforms with respect to $t_1$, we observe that $A$ is replaced by $A+1$. Again in (3.1), replacing $x$ by $x/t_1$, $y$ by $y/t_1$, $z$ by $z/t_1$, multiplying both sides by $(t_1)^{-e_{E+1}}$ and taking the inverse Laplace transform, we find that $E$ is replaced by $E+1$. Thus the induction on $A$ and $E$ is complete. Similarly, induction on other parameters can be performed. Thus the formal proof of (3.1) by induction is completed.

The results (3.2) and (3.3) can be established by the above techniques, making use of the formulae (2.2) and (2.4) respectively.

**4.** *Particular cases of* (3.1), (3.2) *and* (3.3).

It is interesting to observe that the bilinear relations (3.1) to (3.3) can be easily specialised to yield a large number of results involving series of the type

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} \Phi_n(x, y, z)\, \psi_n(\xi, \eta, \rho)\, t^n,$$

where $\Phi_n(x, y, z)$ and $\psi_n(\xi, \eta, \rho)$ are one or the other of the fourteen Lauricella functions $F_1, \ldots, F_{14}$, with the only exception of the function $F_5$ or $F_C^{(3)}$ defined in [8, p. 114]. Thus, we can obtain several bilinear and bilateral generating relations involving these functions of three variables. It is also worthwhile to note that our results (3.1) to (3.3) cannot be reduced to Srivastava's functions $H_A$, $H_B$ and $H_C$.

Some special cases are mentioned below:

(i) We notice that our formulae (3.1) to (3.3) can be easily reduced to the known results of Srivastava ([1, pp. 70—71, (3.1), (3.2) and (3.3)]; see also [3], p. 80).

(ii) With $\xi = \eta = \rho = 0$, (3.1) reduces to

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} F^{(3)}\left[\begin{matrix}(a)::(b);\ (b');\ (b''):\ -n,\ (c);\ (c');\ (c'');\\ (e)::(g);\ (g');\ (g''):\qquad (h);\ (h');\ (h'');\end{matrix}\ x, y, z\right] t^n$$

$$= (1-t)^{-\lambda} F^{(3)}\left[\begin{matrix}(a)::(b);\ (b');\ (b''):\lambda,\ (c);\ (c');\ (c'');\\ (e)::(g);\ (g');\ (g''):\quad (h);\ (h');\ (h'');\end{matrix}\ \frac{xt}{t-1}, y, z\right]. \tag{4.1}$$

Replacing $t$ by $t/\lambda$ and letting $\lambda$ tend to infinity, in above, we obtain

$$\sum_{n=0}^{\infty} F^{(3)}\left[\begin{matrix}(a)::(b);\ (b');\ (b''):\ -n,\ (c);\ (c');\ (c'');\\ (e)::(g);\ (g');\ (g''):\qquad (h);\ (h');\ (h'');\end{matrix}\ x, y, z\right] \frac{t^n}{n!}$$

$$= e^t F^{(3)}\left[\begin{matrix}(a)::(b);\ (b');\ (b''):(c);\ (c');\ (c'');\\ (e)::(g);\ (g');\ (g''):(h);\ (h');\ (h'');\end{matrix}\ -xt, y, z\right]. \tag{4.2}$$

Many interesting special cases of this result can be obtained; one of which is

$$\sum_{n=0}^{\infty} F_D^{(3)}(a, -n, c', c''; e; x, y, z) \frac{t^n}{n!}$$

$$= e^t\, {}_3\Phi_D^{(1)}(a, c', c''; e; y, z, -xt), \tag{4.3}$$

where $F_D^{(3)}$ is the Lauricella function of three variables defined by [8, p. 114] and ${}_3\Phi_D^{(1)}$ is a confluent hypergeometric function of three variables defined by [9, equation (2.9)].

The formulae (4.1) and (4.2) can be reduced on obvious simplification to formulae (7.1) and (7.2) of Srivastava's paper [3, p. 94].

(iii) In (3.1), taking $A-1=E-1=B=B'=B''=C=C'-1=C''-1=G==G'=G''=0$, replacing $\lambda$ by $-\lambda$, $t$, by $-t$, $x$ by $\frac{xy(t+1)}{t}$, $z$ by $yz$, $\xi$ by $\xi\eta\frac{(t+1)}{t}$ and $\rho$ by $\eta\rho$, we obtain a known result due to Deshpande [10, p. 128 (3.4)]. Note that the known results [10, pp. 127—128, (3.1), (3.2), and (3.2)] are contained in the more general bilinear relations (3.1) to (3.3) of Srivastava [1, pp. 70—71].

(iv) In (3.1) putting $y=z=0$, $B=B''=G=G''=C=H=0$ and making use of the result [11, p. 69, ex. 4], we obtain on simplification the interesting formula

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}\,{}_{A+1}F_E\left[\begin{matrix}(a),\ -n;\\ (e);\end{matrix}\ x\right]$$

$$\cdot F^{(3)}\left[\begin{matrix}(u)::(v);\ (v');\ (v''):\lambda+n,\ (w);\ (w');\ (w'');\\ (d)::(k);\ (k');\ (k''):\qquad (l);\ (l');\ (l'');\end{matrix}\ \xi,\ \eta,\ \rho\right]t^n$$

$$=(1-t)^{-\lambda}\sum_{n=0}^{\infty}\frac{(\lambda)_n((a))_n}{n!\ ((e))_n}\left(\frac{xt}{t-1}\right)^n$$

$$\cdot F^{(3)}\left[\begin{matrix}(u)::(v);\ (v');\ (v''):\lambda+n,\ (w);\ (w');\ (w'');\\ (d)::(k);\ (k');\ (k''):\qquad (l);\ (l');\ (l'');\end{matrix}\ \frac{\xi}{1-t},\ \eta,\ \rho\right]. \tag{4.4}$$

The known formula [3, p. 90, (5.11)] will follow as a special case of the above result when $\rho=0$ and $V=K=W=L=W'=L'=0$.

(v) In (3.2) substituting $\eta=\rho=0$, $V=K=V''=K''=W=L=0$, making use of Vandermonde's theorem [11, p. 69, ex. 4] and simplifying, we obtain

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}\,{}_{U+1}F_D\left[\begin{matrix}(u),\ \lambda+n;\\ (d)\end{matrix}\ \xi\right]$$

$$\cdot F^{(3)}\left[\begin{matrix}(a)::(b);\ (b');\ (b''):\ -n,\ (c);\ (c');\ (c'');\\ (e)::(g);\ (g');\ (g''):\qquad (h);\ (h');\ (h'');\end{matrix}\ x,\ y,\ z\right]t^n$$

$$=(1-t)^{-\lambda}\sum_{n=0}^{\infty}\frac{(\lambda)_n((u))_n}{n!\ ((d))_n}\left(\frac{\xi}{1-t}\right)^n$$

$$\cdot F^{(3)}\left[\begin{matrix}(a)::(b);\ (b');\ (b''):\lambda+n,\ (c);\ (c');\ (c'');\\ (e)::(g);\ (g');\ (g''):\qquad (h);\ (h');\ (h'');\end{matrix}\ \frac{xt}{t-1},\ y,\ z\right].$$

In above, substituting $U=D=1$, $u_1=-\alpha$, $d_1=-\alpha-\beta$, $\lambda=m-\alpha-\beta$, replacing $\xi$ by $\frac{2}{\xi+1}$, $t$ by $(1-\xi)\frac{t}{2}$ and employing the formula [3, p. 92, (6.4)]

$$P_{m+n}^{(\alpha-m-n,\ \beta-m-n)}(x)$$

$$=\frac{(-\alpha-\beta)_{m+n}}{(m+n)!}\left(\frac{1-x}{2}\right)^{m+n}\left(\frac{x+1}{x-1}\right)^{\alpha}{}_2F_1\left(\begin{matrix}-\alpha-\beta+m+n,\ -\alpha;\\ -\alpha-\beta;\end{matrix}\ \frac{2}{x+1}\right),$$

$n=0, 1, 2, \ldots,$ and for every non-negative integer $m$, we obtain the formula

$$\sum_{n=0}^{\infty} \binom{m+n}{n} P_{m+n}^{(\alpha-m-n,\ \beta-m-n)}(\xi)$$

$$\cdot F^{(3)}\left[\begin{matrix}(a)::(b);\ (b');\ (b''):\ -n,\ (c);\ (c');\ (c'');\\ (e)::(g);\ (g');\ (g''):\qquad (h);\ (h');\ (h'');\end{matrix}\ x, y, z\right] t^n$$

$$=\left(\frac{1-\xi}{2}\right)^m \left(\frac{\xi+1}{\xi-1}\right)^\alpha \frac{(\sigma)^{\alpha+\beta-m}}{m!} \sum_{n=0}^{\infty} \frac{(-\alpha-\beta)_{m+n}(-\alpha)_n}{n!\,(-\alpha-\beta)_n} \left(\frac{2}{(\xi+1)\sigma}\right)^n$$

$$(4.5)\quad \cdot F^{(3)}\left[\begin{matrix}(a)::(b);\ (b');\ (b''):m+n-\alpha-\beta,\ (c);\ (c');\ (c'');\\ (e)::(g);\ (g');\ (g''):\qquad (h);\ (h');\ (h'');\end{matrix}\ \frac{x(\sigma-1)}{\sigma}, y, z\right].$$

where $\sigma = 1+\frac{1}{2}(\xi-1)\,t$.

The formula (4.5), with $z=0$ and $B=G=C=H=C'=H'=0$, corresponds to Srivastava's result [3, p. 92 (6.5)].

**5.** Srivastava (see [1, p. 70, (3.2)] and [3, p. 80, (3.2)]) has also obtained the following formula:

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} F^{(2)}\left[\begin{matrix}(a)::-n,\ (b);\ (c);\\ (e)::\qquad (g);\ (h);\end{matrix}\ x, y\right] F^{(2)}\left[\begin{matrix}(a'):\lambda+n,\ (b');\ (c');\\ (e')\qquad (g');\ (h');\end{matrix}\ u, v\right] z^n$$

$$=(1-z)^{-\lambda} \sum_{n=0}^{\infty} \frac{(\lambda)_n ((a))_n ((a'))_n ((b))_n ((b'))_n}{n!\ ((e))_n ((e'))_n ((g))_n ((g'))_n} \left[\frac{-xuz}{(1-z)^2}\right]^n$$

$$\cdot F^{(2)}\left[\begin{matrix}(a)+n:\lambda+n,\ (b)+n;\ (c);\\ (e)+n:\qquad (g)+n;\ (h);\end{matrix}\ \frac{xz}{z-1}, y\right]$$

$$(5.1)\quad \cdot F^{(2)}\left[\begin{matrix}(a')+n:\lambda+n,\ (b')+n;\ (c');\\ (e')+n:\qquad (g')+n;\ (h');\end{matrix}\ \frac{u}{1-z}, v\right],$$

where $F^{(2)}[x, y]$ denotes Kampé de Fériet's double hypergeometric function [8, p.150] in the contracted notation of Burchnall and Chaundy [12, p. 112].

We observe that the right-hand side of (5.1) can be simplified by using Vandermonde's theorem [11, p. 69, ex. 4] as

$$(1-z)^{-\lambda} \sum_{n=0}^{\infty} \frac{((a'))_n ((c'))_n v^n}{n!\ ((e)')_n ((h'))_n}$$

$$(5.2)\quad \cdot F^{(3)}\left[\begin{matrix}\text{---}::\lambda;\ \text{---};\ (a):(b);\ (b'),\ (a')+n;\ (c);\\ \text{---}::\text{---};\ \text{---};\ (e):(g);\ (g'),\ (e')+n;\ (h);\end{matrix}\ \frac{xz}{z-1}, \frac{u}{1-z}, y\right].$$

Proceeding on the same lines as above, we find that the right-hand side of the formula (3.2) of this paper can also be written as

$$(1-t)^{-\lambda} \sum_{n=0}^{\infty} \frac{(\lambda)_n ((u))_n ((v))_n ((v''))_n ((w))_n}{n! \; ((d))_n ((k))_n ((k''))_n ((l))_n} \left(\frac{\xi}{1-t}\right)^n$$

$$\cdot F^{(2)} \begin{bmatrix} (u)+n, & (v'):(v)+n, & (w'); & (v'')+n; & (w''); & \eta, \rho \\ (d)+n, & (k'):(k)+n, & (l); & (k'')+n; & (l''); & \end{bmatrix}$$

$$\cdot F^{(3)} \begin{bmatrix} (a)::(b); & (b'); & (b''):\lambda+n, & (c); & (c'); & (c''); & \frac{xt}{t-1}, y, z \\ (e)::(g); & (g'); & (g''): & (h); & (h'); & (h''); & \end{bmatrix}. \tag{5.3}$$

Thus, we can obtain these results in more compact forms.

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