PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 17 (31), 1974, pp. 52—58

ON SOME MAPS WITH A NONUNIQUE FIXED POINT

Ljubomir B. Ćirić

(Received April 4, 1974)

1. Introduction.

Let (M, d) be a metric space and let T be a function of M into itself. A mapping T is called a *contraction* if there exists a positive real number $q<1$ such that

$$d(Tx, Ty) \leqslant q \cdot d(x, y)$$

holds for all $x, y \in M$. If M is a complete space, then, by Banach fixed point principle, T has a unique fixed point in M.

In [1] we introduced quasi-contractions by requiring

$$d(Tx, Ty) \leqslant q \cdot \max\{d(x, y),\ d(x, Tx),\ d(y, Ty),\ d(x, Ty),\ d(y, Tx)\}.$$

We showed that Banach's result for contractions can be extended to quasi-contractions.

The purpose of this paper is to consider functions T on M which are not necessarily continuous and which satisfy a condition of the type

$$\min\{d(Tx, Ty),\ d(x, Tx),\ d(y, Ty)\} - \min\{d(x, Ty),\ d(y, Tx)\} \leqslant q \cdot d(x, y).$$

Various theorems about fixed and periodic points are obtained. Multi-valued functions are also discussed.

2. Single-valued maps

We recall that a mapping T on M is *orbitally continuous* if $\lim_i T^{n_i} x = u$ implies $\lim_i TT^{n_i} x = Tu$ for each $x \in M$. A space M is *T-orbitally complete* if every Cauchy sequence of the form $\{T^{n_i} x\}_{i=1}^{\infty}$, $x \in M$, converges in M (cf. [4]).

Theorem 1. *Let $T: M \to M$ be an orbitally continuous mapping on M and let M be T-orbitally complete. If T satisfies the following condition*

(1) $\min\{d(Tx, Ty),\ d(x, Tx),\ d(y, Ty)\} - \min\{d(x, Ty),\ d(y, Tx)\} \leqslant q \cdot d(x, y)$

for some $q<1$ and all $x, y \in M$, then for each $x \in M$, the sequence $\{T^n x\}_{n=1}^{\infty}$ converges to a fixed point of T.

Proof. Let $x \in M$ be arbitrary. We shall show that the sequence of iterates

(2) $$x_0 = x, \quad x_1 = Tx_0, \quad x_2 = Tx_1, \; \dots, \; x_n = Tx_{n-1}, \; \dots$$

at x is a Cauchy's sequence. Since $x_{k-1} = x_k$ for some $k \in I^+$ (I^+- the positive integers) immediately implies that $\{x_n\}$ is the Cauchy's sequence, we may suppose that $x_{n-1} \neq x_n$ for each $n = 1, 2, \dots$ By (1) for $x = x_{n-1}$ and $y = x_n$ we have

$$\min\{d(Tx_{n-1}, Tx_n),\; d(x_{n-1}, Tx_{n-1}),\; d(x_n, Tx_n)\} -$$
$$-\min\{d(x_{n-1}, Tx_n),\; d(x_n, Tx_{n-1})\} = \min\{d(x_n, x_{n+1}),\; d(x_{n-1}, x_n)\} \leq$$
$$\leq q \cdot d(x_{n-1}, x_n).$$

Since $d(x_{n-1}, x_n) \leq qd(x_{n-1}, x_n)$ is impossible (as $q < 1$), one has

$$d(x_n, x_{n+1}) \leq q \cdot d(x_{n-1}, x_n).$$

Proceeding in this manner we obtain

$$d(x_n, x_{n+1}) \leq qd(x_{n-1}, x_n) \leq q^2 d(x_{n-2}, x_{n-1}) \leq \cdots \leq q^n d(x, Tx).$$

Hence for any $p \in I^+$ one has

$$d(x_n, x_{n+p}) \leq \sum_{k=n}^{n+p-1} d(x_k, x_{k+1}) \leq \left(\sum_{k=n}^{n+p-1} q^k\right) d(x, Tx) \leq \frac{q^n}{1-q} d(x, Tx).$$

Since $\lim_n q^n = 0$ it follows that (2) is the Cauchy's sequence. M being T-orbitally complete, there is some $u \in M$ such that $u = \lim_n T^n x$. By orbital continuity of T

$$Tu = \lim_n TT^n x = u,$$

i.e. u is a fixed point of T.

The proof of the Theorem is complete.

If T is not orbitally continuous, then T may fail to have a fixed point, even if M is compact and connected. For example, let $M = [0, 1]$ be a subset of reals with the usual metric. Define T on M by $Tx = \frac{2}{3}x$, if $x \neq 0$ and $T(0) = 1$.Then T satisfies (1) with $q = \frac{2}{3}$, but T is not orbitally continuous and has no fixed point.

If we now define a discontinuous function T_1 on [0, 1] as follows

$$T_1 x = \frac{2}{3}x, \text{ if } x \text{ rational}$$
$$= x, \text{ if } x \text{ irrational},$$

then T_1 is orbitally continuous, satisfies (1) with $q = \frac{2}{3}$ and has infinitely many fixed points.

Theorem 2. *Let $T: M \to M$ be an orbitally continuous mapping of a T-orbitally complete metric space M into itself and let $\varepsilon > 0$. If there exists a point x_0 in M such that $d(x_0, T^k x_0) < \varepsilon$ for some $k \in I^+$ and if T satisfies*

(3) $0 < d(x, y) < \varepsilon$ *implies* $\min\{d(Tx, Ty),\ d(x, Tx),\ d(y, Ty)\} \leqslant q d(x,y)$

for some $q < 1$ and all $x, y \in M$, then T has a periodic point.

Proof. A subset $K = \{k : d(x, T^k x) < \varepsilon$ for some $x \in M\}$ of the set I^+ is non-void by assumption of Theorem. Denote $m = \min K$ and let $x \in M$ be such that $d(x, T^m x) < \varepsilon$.

If $m = 1$ then for x and Tx by (3)

$$\min\{d(Tx, TTx),\ d(x, Tx),\ d(Tx, TTx)\} \leqslant q d(x, Tx)$$

and hence

$$d(Tx, T^2 x) \leqslant q d(x, Tx) < q\varepsilon.$$

Proceeding as in Theorem 1. we obtain that $Tu = u$ for some $u \in M$.

Suppose now that $m \geqslant 2$, i.e. that

$$d(y, Ty) \geqslant \varepsilon \tag{4}$$

for each $y \in M$. Then from $d(x, T^m x) < \varepsilon$ and by (3) it follows

$$\min\{d(Tx, T^{m+1} x),\ d(x, Tx),\ d(T^m x, T^{m+1} x)\} \leqslant q \cdot d(x, T^m x) < \varepsilon.$$

Since by (4) $d(x, Tx) \geqslant \varepsilon$ and $d(T^m x, T^{m+1} x) = d(T^m x, TT^m x) \geqslant \varepsilon$, one has

$$d(Tx, T^{m+1} x) \leqslant q \cdot d(x, T^m x) < q\varepsilon.$$

By the same reason

$$d(T^2 x, T^{m+2} x) \leqslant q \cdot d(Tx, T^{m+1} x) < q^2 \varepsilon.$$

Proceeding in this manner we obtain

$$d(T^n x, T^{m+n} x) \leqslant q \cdot d(T^{n-1} x, T^{m+n-1} x) \leqslant \cdots \leqslant q^n d(x, T^m x) < q^n \varepsilon,$$

for each $n \in I^+$. Thence, for the sequence

$$x_0 = x.\quad x_1 = T^m x_0,\quad x_2 = T^m x_1, \ \ldots\ ,\ x_{n+1} = T^m x_n, \ \ldots$$

we have that

$$d(x_n, x_{n+1}) = d(T^{nm} x, T^{m+nm} x) \leqslant q^{nm} d(x, T^m x) < q^{nm} \varepsilon.$$

Then, by routine calculation, it follows that $\{x_n\}$ is a Cauchy's sequence. As $\{x_n\} \subseteq \{T^n x\}$ and M is T-orbitally complete, there exists some $u \in M$ such that $u = \lim_n x_n = \lim_n T^{nm} x$. Since orbital continuity of T implies orbital continuity of T^s for any $s \in I^+$, as can be easily verified, one has

$$T^m u = \lim_n T^m T^{nm} x = \lim_n T^{(n+1)m} x = u,$$

which completes the proof of the Theorem.

Now we shall consider sufficient conditions for the existence of fixed and periodic points for maps which satisfy (1) or (3) with $q = 1$. Such conditions are always satisfied in compact spaces.

Theorem 3. *Let $T: M \to M$ be an orbitally continuous mapping on a metric space M which satisfies the following condition*

(5) $\min\{d(Tx, Ty),\ d(x, Tx),\ d(y, Ty)\} - \min\{d(x, Ty),\ d(y, Tx)\} < d(x, y)$

whenever $x \neq y$. If for some $x_0 \in M$ the sequence $\{T^n x_0\}$ has a cluster point $u \in M$, then u is a fixed point of T.

Proof. If $T^{r-1} x_0 = T^r x_0$ for some $r \in I^+$, then $T^n x_0 = T^r x_0 = u$ for all $n \geq r$, and assertion follows. Assume now that $T^{r-1} x_0 \neq T^r x_0$ for all $r \in I^+$, and let $\lim_i T^{n_i} x_0 = u$. Then for $T^{n-1} x_0,\ T^n x_0 \in M$, by (5),

$$\min\{d(TT^{n-1} x_0,\ TT^n x_0),\ d(T^{n-1} x_0,\ T^n x_0),\ d(T^n x_0,\ TT^n x_0)\} -$$
$$-\min\{d(T^{n-1} x_0,\ T^{n+1} x_0),\ 0\} = \min\{d(T^n x_0,\ T^{n+1} x_0),\ d(T^{n-1} x_0,\ T^n x_0)\} <$$
$$< d(T^{n-1} x_0,\ T^n x_0).$$

Hence $d(T^n x_0,\ T^{n+1} x_0) < d(T^{n-1} x_0,\ T^n x_0)$, as $d(T^{n-1} x_0,\ T^n x_0) < d(T^{n-1} x_0,\ T^n x_0)$ is impossible. Therefore, $\{d(T^n x_0,\ T^{n+1} x_0)\}_{n \in I^+}$ is a decreasing and hence convergent sequence of positive reals. Since $\lim_i d(T^{n_i} x_0,\ T^{n_i+1} x_0) = d(u, Tu)$ and $\{d(T^{n_i} x_0,\ T^{n_i+1} x_0)\} \subseteq \{d(T^n x_0,\ T^{n+1} x_0)\}$, it follows that

(6) $$\lim_n d(T^n x_0,\ T^{n+1} x_0) = d(u, Tu).$$

Also, as $\lim_i T^{n_i+1} x_0 = Tu$, $\lim_i T^{n_i+2} x_0 = T^2 u$ and

$$\{d(T^{n_i+1} x_0,\ T^{n_i+2} x_0)\} \subseteq \{d(T^n x_0,\ T^{n+1} x_0)\},$$

by (6)

(7) $$d(Tu, T^2 u) = d(u, Tu).$$

Suppose that $d(u, Tu) > 0$. Then by (5)

$$d(Tu, T^2 u) < d(u, Tu),$$

which contradicts (7). This proves that $Tu = u$.

Theorem 4. *Let $T: M \to M$ be an orbitally continuous map on M and let $\varepsilon > 0$. If T satisfies the following condition*

(8) $0 < d(x, y) < \varepsilon$ *implies* $\min\{d(Tx, Ty),\ d(x, Tx),\ d(y, Ty)\} < d(x, y)$

and if for some $x_0 \in M$ the sequence $\{T^n x_0\}_{n \in I^+}$ has a cluster point $u \in M$, the n u is a point of T.

Proof. Let $\lim_i T^{n_i} x_0 = u$. Then there exists $N_1 \in I^+$ with the property that $i > N_1$ implies $d(T^{n_i} x_0, u) < \frac{\varepsilon}{2}$ Hence $d(T^{n_i} x_0,\ T^{n_i+1} x_0) < \varepsilon$ and a set

$$K = \{k \in I^+ : d(T^r x_0,\ T^{r+k} x_0) < \varepsilon \text{ for some } r \in I^+\}$$

is non-void. Put $m = \min K$. If $d(T^s x_0,\ T^{s+m} x_0) = 0$ for some $s \in I^+$, then $u = T^s x_0 = T^m u$ and assertion follows.

Suppose that $d(T^s x_0, T^{s+m} x_0) > 0$ for every $s \in I^+$ and let $r \in I^+$ be such that $d(T^r x_0, T^{r+m} x_0) < \varepsilon$.

If $m = 1$, then by (8), as in the proof of Theorem 3, $\{d(T^n x_0, T^{n+1} x_0)\}_{n \in I^+}$ decreases for $n \geqslant r$ and it follows that $Tu = u$.

Suppose now that $m \geqslant 2$, i.e. that

$$d(T^n x_0, T^{n+1} x_0) \geqslant \varepsilon \tag{9}$$

for every $n \in I^+$. Since, by orbital continuity of T, $\lim_i T^{n_i+s} x_0 = T^s u$, we have by (9)

$$d(T^s u, T^{s+1} u) = \lim_i d(T^{n_i+s} x_0, T^{n_i+s+1} x_0) \geqslant \varepsilon, \tag{10}$$

for every $s \in I^+$.

By (8) and the assumption $0 < d(T^r x_0, T^{r+m} x_0) < \varepsilon$ we have

$$\min \{d(T^{r+1} x_0, T^{r+m+1} x_0),\ d(T^r x_0, T^{r+1} x_0),\ d(T^{r+m} x_0, T^{r+m+1} x_0)\}$$
$$< d(T^r x_0, T^{r+m} x_0).$$

Hence and by (9)

$$d(T^{r+1} x_0, T^{r+1+m} x_0) < d(T^r x_0, T^{r+m} x_0) < \varepsilon.$$

By the same reason we obtain the following inequalities

$$\varepsilon > d(T^r x_0, T^{r+m} x_0) > d(T^{r+1} x_0, T^{r+1+m} x_0) > d(T^{r+2} x_0, T^{r+2+m} x_0) > \cdots. \tag{11}$$

Hence the sequence $\{d(T^n x_0, T^{n+m} x_0 : n \geqslant r\}$ is decreasing and, therefore, is convergent. Since its subsequences $\{d(T^{n_i} x_0, T^{n_i+m} x_0) : i \in I^+\}$ and $\{d(T^{n_i+1} x_0, T^{n_i+1+m} x_0) : i \in I^+\}$ converge to $d(u, T^m u)$ and $d(Tu, T^{m+1} u)$ respectively (by orbital continuity of T and $\lim_i T^{n_i} x_0 = u$), it follows that

$$d(Tu, T^{m+1} u) = d(u, T^m u) = \lim_n d(T^n x_0, T^{n+m} x_0). \tag{12}$$

Now we shall show that $T^m u = u$. Since from (11) and (12) it follows that $d(u, T^m u) < \varepsilon$, then by (8), if we suppose that $d(u, T^m u) > 0$,

$$\min \{d(Tu, T^{m+1} u),\ d(u, Tu),\ d(T^m u, T^{m+1} u)\} < d(u, T^m u) < \varepsilon.$$

Hence we obtain, using (10),

$$d(Tu, T^{m+1} u) < d(u, T^m u),$$

which is clearly a contradiction with (12). This completes the proof of the Theorem.

3. Multi-valued functions

Let (M, d) be a generalized metric space (i.e. a pair (M, d) where M is a set and $d : M \times M \to [0, \infty]$ satisfies all the properties of being a metric for M exept that d may have "infinite values"). A multi-valued function $F : M \to M$ is a point-to-set correspondence. An *orbit* of F at the point $x \in M$ is a sequence $\{x_n : x_n \in F x_{n-1}\}$, where $x_0 = x$. A multi-valued function F is *orbitally upper-semicontinuous* if $x_n \to u \in M$ implies $u \in Fu$ whenever $\{x_n\}$ is an orbit of F at some $x \in M$. A space M is *F-orbitally complete* if every orbit of F at some $x \in M$ which is Cauchy sequence, converges in M. Let A and B be nonempty subsets of M. Denote

$$D(A, B) = \inf\{d(a, b) : a \in A, \quad b \in B\},$$

$$CL(M) = \{A : A \text{ is closed}\},$$

$$N(\varepsilon, A) = \{x \in M : d(x, a) < \varepsilon \text{ for some } a \in A\}, \quad \varepsilon > 0$$

$$H(A, B) = \begin{cases} \inf\{\varepsilon > 0 : A \subseteq N(\varepsilon, B) \text{ and } B \subseteq N(\varepsilon, A) \text{ if the infimum exists} \\ \infty, \text{ otherwise.} \end{cases}$$

We are now ready to state the following result.

Theorem 5. *Let (M, d) be a generalized metric space and let $F: M \to CL(M)$ be orbitally upper-semicontinuous. If M is F-orbitally complete and F satisfies the following condition*

$$\text{(13)} \quad \min\{H(Fx, Fy), \ D(x, Fx), \ D(y, Fy)\} - \min\{D(x, Fy), \ D(y, Fx)\} \leqslant \leqslant q \cdot d(x, y)$$

for some $q < 1$ and all $x, y \in M$, then F has a fixed point.

Proof. Let $a > 0$ be an arbitrary small real number less than 1. We define a single-valued function $T: M \to M$ by letting Tx to be a point $y \in Fx$ that satisfies $d(x, y) \leqslant q^{-a} D(x, Fx)$.

Let now x be arbitrary point in M and let us consider the following orbit of F at x

$$\text{(14)} \qquad x_0 = x, \quad x_1 = Tx_0, \quad x_2 = Tx_1, \ \ldots, \ x_n = Tx_{n-1}, \ \ldots$$

We may assume that $x_{n-1} \neq x_n$ for any $n \in I^+$, since otherwise the assertion of the Theorem follows at once. Since

$$x_n \in Fx_{n-1} \text{ implies } D(x_n, Fx_n) \leqslant H(Fx_{n-1}, Fx_n) \text{ and } D(x_n, Fx_{n-1}) = 0$$

we have

$$\min\{H(Fx_{n-1}, Fx_n), \ D(x_{n-1}, Fx_{n-1}), \ D(x_n, Fx_n)\} - \\ - \min\{D(x_{n-1}, Fx_n), \ D(x_n, Fx_{n-1})\} = \min\{D(x_{n-1}, Fx_{n-1}), \ D(x_n, Fx_n)\}$$

and by (13)

$$\min\{q^{-a} D(x_{n-1}, Fx_{n-1}), \ q^{-a} D(x_n, Fx_n)\} \leqslant q^{-a} q d(x_{n-1}, x_n).$$

Thence and by definition of the function T and the sequence (14) it follows that

$$\min\{d(x_{n-1}, Tx_{n-1}), \ d(x_n, Tx_n)\} \leqslant q^{1-a} d(x_{n-1}, x_n),$$

and hence

$$\text{(15)} \qquad d(x_n, x_{n+1}) \leqslant q^{1-a} d(x_{n-1}, x_n).$$

Since $q^{1-a} < 1$, by routine calculation from (15) it follows that the orbit (14) of F at x is a Cauchy sequence. Being M F-orbitally complete, there is some $u \in M$ such that

$$\lim_n x_n = u.$$

Then orbital upper-semicontinuity of F implies $Fu = u$, which completes the proof of the Theorem.

REFERENCES

[1] Ćirić Lj., *A generalization of Banach's contraction principle*, Proc. Amer. Math. Soc. (to appear).

[2] Ćirić Lj., *Generalized contractions and fixed-point theorems*, Publ. Inst. Math., T. 12 (26), 1971, 19—26.

[3] Ćirić Lj., *Fixed points for generalized multi-valued contractions*, Mat. vesnik 9 (24), Sv. 3, 1972, 265—272.

[4] Ćirić Lj., *On contraction type mappings*, Math. Balkanica 1, 1971, 52—57.

[5] Kurepa Đ., *Some fixed point theorems*, Math. Balcan., 2 (1972), pp. 101—108.