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ON A FAMILY OF CONTRACTIVE MAPS AND FIXED POINTS

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1. Introduction.

Let (M, d) be a metric space and let $\mathcal{F}=\{T_\lambda : \lambda \in (\lambda)\}$ be a family of maps which map M into itself. A point $u \in M$ is a common fixed point for $\mathcal{F}$ iff $u=T_\lambda u$ for each $T_\lambda \in \mathcal{F}$. A mapping $T: M \to M$ is called a generalized contraction iff

$$(1)\quad d(Tx, Ty) \leqslant q \cdot \max\left\{d(x, y);\ d(x, Tx);\ d(y, Ty);\ \frac{1}{2}[d(x, Ty)+d(y, Tx)]\right\}$$

holds for some $q<1$ and all $x, y \in M$. M is T-orbitally complete iff every Cauchy sequence of the form $\{T^{n_i}x\}_{i \in N}$, $x \in M$, converges in M. In [2] we proved the following result.

Theorem A. *A generalized contraction T on T-orbitally complete metric space M has a unique fixed point.*

In a recent paper [1] S. K. Chatterjea proved the following:

Theorem B. *If there exists a sequence of continuous mappings $\{T_n\}$ of M into itself such that for some m and $0<q<1$*

(i) *for every T_i and T_j $d(T_i^m x, T_j^m y) \leqslant q \cdot d(x, y)$, $x, y \in M$,*

(ii) *T_i commutes with T_j, $i \neq j$,*

then $\{T_n\}$ has a unique common fixed point.

In this paper we investigate a family of maps which satisfy a common condition of type (1) and which are not necessarily continuous and commuting. An example is given to show that our results are indeed extension of Theorem B.

2. Let S be a set and $T: S \to S$ be a map of S in S. Denote $F(T)=\{x \in S : x=Tx\}$.

Lemma. *Let T_0, $T: M \to M$ be two maps on a metric space (M, d). If*

$$(2)\quad d(T_0 x, Ty) \leqslant q \cdot \max\{d(x, y),\ d(x, T_0 x),\ d(y, Ty),\ d(x, Ty),\ d(y, T_0 x)\}$$

holds for some $q<1$ and all $x, y \in M$, and $F(T_0)$ is a non empty set, then $F(T_0)$ is a singleton and $F(T)=F(T_0)$.

Proof. Let $u \in F(T_0) \subset M$ be any fixed point. Then by (2)

$$d(u, Tu) = d(T_0 u, Tu) \leq q \cdot \max\{d(u, u),\ d(u, T_0 u),\ d(u, Tu),\ d(u, Tu),\ d(u, T_0 u)\} = q \cdot d(u, Tu),$$

and hence $d(u, Tu) \cdot (1-q) \leq 0$, which implies $d(u, Tu) = 0$. Therefore, $u \in F(T)$. Let now $v \in F(T_0)$ be arbitrary. Then $v \in F(T)$ and by (2)

$$d(u, v) = d(T_0 u, Tv) \leq q \cdot \max\{d(u, v),\ 0,\ 0,\ d(u, v),\ d(u, v)\} = q \cdot d(u, v).$$

Thence $v = u$. Therefore, $F(T_0) = \{u\} = F(T)$.

Now we shall use Lemma to prove the following results:

Theorem 1. *Let $\{T_n : n \in I^+\}$ be a sequence of maps on a complete metric space (M, d). If for some $q \in (0, 1) \subset R$*

$$(3) \qquad d(T_0 x, T_n y) \leq q \cdot \max\left\{d(x, y),\ d(x, T_0 x),\ d(y, T_n y),\ \frac{1}{2}[d(x, T_n y) + d(y, T_0 x)]\right\}$$

holds for each $n = 1, 2, \ldots$ and all $x, y \in M$, then there exists a unique point $u \in M$ such that $T_n u = u$ for each $n = 0, 1, 2, \ldots$ and for arbitrary $x_0 \in M$ a sequence

$$(4) \qquad x_0,\ x_1 = T_0 x_0,\ x_2 = T_1 x_1,\ x_3 = T_0 x_2,\ \ldots,\ x_{2n-1} = T_0 x_{2n-2},\ x_{2n} = T_n x_{2n-1},\ \ldots$$

converges to u.

Proof. We prove that (4) is a Cauchy sequence, where $x_0 \in M$ is arbitrary. By (3) for $x = x_{2n-2}$ and $y = x_{2n-1}$

$$d(x_{2n-1}, x_{2n}) = d(T_0 x_{2n-2}, T_n x_{2n-1})$$

$$\leq q \cdot \max\{d(x_{2n-2}, x_{2n-1}),\ d(x_{2n-2}, T_0 x_{2n-2}),\ d(x_{2n-1}, T_n x_{2n-1}),$$

$$\frac{1}{2}[d(x_{2n-2}, T_n x_{2n-1}) + d(x_{2n-1}, T_0 x_{2n-2})] =$$

$$= q \cdot \max\left\{d(x_{2n-2}, x_{2n-1}),\ d(x_{2n-1}, x_{2n}),\ \frac{1}{2} d(x_{2n-2}, x_{2n})\right\}.$$

Since

$$d(x_{2n-1}, x_{2n}) \leq q \cdot d(x_{2n-1}, x_{2n}) \text{ implies } d(x_{2n-1}, x_{2n}) = 0$$

and

$$d(x_{2n-1}, x_{2n}) \leq q \cdot \frac{1}{2} d(x_{2n-2}, x_{2n}) \text{ implies } \frac{1}{2} d(x_{2n-2}, x_{2n}) \leq d(x_{2n-2}, x_{2n-1}),$$

we have

$$d(x_{2n-1}, x_{2n}) \leq q \cdot d(x_{2n-2}, x_{2n-1}).$$

By the same reason

$$d(x_{2n-2}, x_{2n-1}) = d(T_{n-1} x_{2n-3}, T_0 x_{2n-2}) \leq q \cdot d(x_{2n-3}, x_{2n-2}).$$

Proceeding in this manner one has

$$d(x_{2n-1}, x_{2n}) \leqslant q \cdot d(x_{2n-2}, x_{2n-1}) \leqslant q^2 (x_{2n-3}, x_{2n-2}) \leqslant \cdots \leqslant q^{2n-1} d(x_0, x_1).$$

By routine calculation one can show that the following inequalities hold

$$d(x_i, x_j) \leqslant \sum_{k=i}^{j-1} d(x_k, x_{k+1}) \leqslant q^i \frac{d(x_0, x_1)}{1-q}; \quad j>i.$$

Therefore, (4) is a Cauchy sequence. Then completeness of M implies that for some $u \in M$

$$\lim_n x_n = u. \tag{5}$$

Using (3) and the triangle inequality we have

$$d(u, T_0 u) \leqslant d(u, x_{2n}) + d(T_n x_{2n-1}, T_0 u) \leqslant d(u, x_{2n}) +$$

$$+ q \cdot \max \left\{ d(x_{2n-1}, u),\ d(x_{2n-1}, x_{2n}),\ d(u, T_0 u),\ \frac{1}{2}[d(x_{2n-1}, T_0 u) + d(u, x_{2n})] \right\}.$$

Hence, as

$$\frac{1}{2} d(x_{2n-1}, T_0 u) \leqslant d(x_{2n-1}, T_0 u) \leqslant d(x_{2n-1}, u) + d(u, T_0 u),$$

we have

$$d(u, T_0 u) \leqslant d(u, x_{2n}) + q \{ d(x_{2n-1}, u) + d(x_{2n-1}, x_{2n}) + d(u, T_0 u) + d(u, x_{2n})] \}.$$

Thence

$$d(u, T_0 u) \leqslant \frac{1}{1-q} [(1+q) d(u, x_{2n}) + q \cdot d(x_{2n-1}, u) + q \cdot d(x_{2n-1}, x_{2n})].$$

This implies by (5), that $d(u, T_0 u) = 0$. Since (3) implies (2), by our Lemma u is a unique fixed point of T_0 and $T_n u = u$ for each $n = 1, 2, \ldots$. This completes the proof of the Theorem.

Theorem 2. *Let $\mathcal{F} = \{T_\lambda : \lambda \in (\lambda)\}$ be a family of functions which maps a complete metric space (M, d) into itself and let $0 \leqslant q < 1$. If there exists some $T_{\lambda_0} \in \mathcal{F}$ such that for each $T_\lambda \in \mathcal{F}$ $(\lambda \neq \lambda_0)$ there are positive integers i_λ and j_λ such that*

$$d(T_{\lambda_0}^{i_\lambda} x, T_\lambda^{j_\lambda} y) \leqslant q \cdot \max \left\{ d(x, y),\ d(x, T_{\lambda_0}^{i_\lambda} x),\ d(y, T_\lambda^{j_\lambda} y),\ \frac{1}{2}[d(x, T_\lambda^{j_\lambda} y) + d(y, T_{\lambda_0}^{i_\lambda} x)] \right\} \tag{6}$$

holds for all $x, y \in M$, then every $T_\lambda \in \mathcal{F}$ has a unique fixed point in M, which is a unique common fixed point for $\mathcal{F}$.

Proof. Let $T_\lambda \in \mathcal{F}$ be arbitrary. For arbitrary $x \in M$ let us consider a sequence

$$x_0 = x,\ x_1 = T_{\lambda_0}^{i_\lambda} x_0,\ x_2 = T_\lambda^{j_\lambda} x_1, \ldots,\ x_{2n-1} = T_{\lambda_0}^{i_\lambda} x_{2n-2},\ x_{2n} = T_\lambda^{j_\lambda} x_{2n-1}, \ldots . \tag{7}$$

By (6)

$$d(x_{2n}, x_{2n+1}) = d(T_{\lambda}^{j\lambda} x_{2n-1}, T_{\lambda_0}^{i\lambda} x_{2n})$$

$$\leqslant q \cdot \max \Big\{ d(x_{2n-1}, x_{2n}),\ d(x_{2n-1}, T_{\lambda}^{j\lambda} x_{2n-1}),\ d(x_{2n}, T_{\lambda_0}^{i\lambda} x_{2n}),$$

$$\frac{1}{2} [d(x_{2n-1}, T_{\lambda_0}^{i\lambda} x_{2n}) + d(x_{2n}, T_{\lambda}^{j\lambda} x_{2n-1})] \Big\}$$

$$= q \cdot \max \Big\{ d(x_{2n-1}, x_{2n}),\ d(x_{2n}, x_{2n+1}),\ \frac{1}{2} \cdot d(x_{2n-1}, x_{2n+1})] \Big\}.$$

Hence, as in the part of proof of Theorem 1,

$$d(x_{2n}, x_{2n+1}) \leqslant q \cdot d(x_{2n-1}, x_{2n}).$$

Then by routine calculation one can show that (7) is the Cauchy sequence. Using completeness of M we have that

$$\lim_n x_n = u$$

for some $u \in M$. By (6)

$$d(u, T_{\lambda_0}^{i\lambda} u) \leqslant d(u, x_{2n}) + d(T_{\lambda}^{j\lambda} x_{2n-1}, T_{\lambda_0}^{i\lambda} u) \leqslant d(u, x_{2n}) +$$

$$+ q \cdot \max \Big\{ d(x_{2n-1}, u),\ d(x_{2n-1}, x_{2n}),\ d(u, T_{\lambda_0}^{i\lambda} u),\ \frac{1}{2} [d(x_{2n-1}, T_{\lambda_0}^{i\lambda} u) + d(u, x_{2n})] \Big\}.$$

Hence

$$d(u, T_{\lambda_0}^{i\lambda} u) \leqslant \frac{1}{1-q} [(1+q)\, d(u, x_{2n}) + q \cdot d(u, x_{2n-1}) + q \cdot d(x_{2n-1}, x_{2n})].$$

Then, as $\lim_k x_k = u$, we have $d(u, T_0 u) = 0$. Therefore, u is a fixed point of $T_{\lambda_0}^{i\lambda}$.

By Lemma u is a unique fixed point of $T_{\lambda_0}^{i\lambda}$ and $T_{\lambda}^{j\lambda}$, as (6) implies (2). Since

$$T_{\lambda_0}^{i\lambda} (T_{\lambda_0} u) = T_{\lambda_0} (T_{\lambda_0}^{i\lambda} u) = T_{\lambda_0} u,$$

$T_{\lambda_0} u$ is also a fixed point of $T_{\lambda_0}^{i\lambda}$ and therefore $T_{\lambda_0} u = u$. Similarly follows that $T_{\lambda} u = u$. So we proved that u is a unique fixed point of T_{λ_0} and T_{λ}.

Now we shall show that u is a unique common fixed point for $\mathcal{F}$. Let $T_{\lambda'} \in \mathcal{F}$, $\lambda_0 \neq \lambda' \neq \lambda$, be arbitrary. Since $u = T_{\lambda_0} u$ implies $u = T_{\lambda_0}^{i\lambda'} u$, by (6) and Lemma, u is a unique fixed point of $T_{\lambda'}^{j\lambda'}$. This implies that u is a unique fixed point of $T_{\lambda'}$. This completes the proof of the Theorem.

Note that the Theorem 2. also includes as a special case Theorem B and the following result of S. K. Chatterjea [1].

Theorem C. *If there exists a sequence of mappings $\{T_n\}$ of a complete metric space (M, d) into itself such that for any two mappings T_i, T_j we have*

1) $d(T_i^m x, T_j^m y) \leq q d(x, y)$

2) $d(T_i^m x \ T_j y) \leq q d(x, y)$

for some m and $0<q<1$; $x, y \in M$, then $\{T_n\}$ has a unique common fixed point.

Now we give an example of a family of maps satisfying the conditions of Theorem 1, for which the conditions of Theorem B and Theorem C did not hold.

Example. Let $M=[0, 1]$ be the subset of reals with the usual metric and let $\mathcal{F}=\{T_0, T_1, \ldots, T_n, \ldots\}$ be a family of functions which maps M into itself, defined as follows

$$T_0 x = \frac{1}{5} x^2, \quad \text{if } x \text{ rational,}$$

$$= \frac{1}{6} x^2, \quad \text{if } x \text{ irrational}$$

and

$$T_n x = \frac{n}{1+5n} x^2, \quad \text{if } x \text{ rational,}$$

$$= \frac{n}{1+6n} x^2, \quad \text{if } x \text{ irrational,} \quad n=1, 2, \ldots .$$

Let $x,y \in M$ and $T_n \in \mathcal{F}$ be arbitrary. If, for example, $T_0 x < T_n y$, then

$$d(T_0 x, T_n y) \leq \frac{1}{4} d(T_0 x, y) \leq \frac{1}{2} \cdot \frac{1}{2} [d(T_0 x, y) + d(y, T_n x)].$$

The case $T_n y < T_0 x$ is now obvious. So we see that (3) is satisfied with $q=\frac{1}{2}$. The point $u=0$ is the unique common fixed point for $\mathcal{F}$. But it is clear that every $T_i \in \mathcal{F}$ is not continuous and that $T_i T_j x \neq T_j T_i x$ for $x \neq 0$ and $i \neq j$.

Theorem 3. *Let M be a complete metric space and let $\{T_n\}$ be a sequence of functions which map M into itself. If there exists some $q\,(0<q<1)$ and a convergent series $\sum_{k=1}^{\infty} a_k \,(a_k \geq 0)$ such that*

$$(8) \qquad d(T_n x, T_{n+1} y) \leq q \cdot \max \Big\{ d(x, y);\ d(x, T_n x);\ d(y, T_{n+1} y);\ \frac{1}{2}[d(x, T_{n+1} y) + d(y, T_n x)] \Big\} + a_n$$

holds for every $x, y \in M$ and each $n=1, 2, \ldots$, then there exists a mapping $T: M \to M$ defined by $Tx = \lim_n T_n x$ which has a unique fixed point in M.

P r o o f. Let $x \in M$ be arbitrary and consider a sequence

$$x_0 = x; \quad x_1 = T_1 x_0, \ x_2 = T_2 x_1, \ \dots, \ x_n = T_n x_{n-1}, \ \dots .$$

By (8)

$$d(x_n, x_{n+1}) = d(T_n x_{n-1}, T_{n+1} x_n)$$

$$\leqslant q \cdot \max \left\{ d(x_{n-1}, x_n); \ d(x_n, x_{n+1}); \ \frac{1}{2} \cdot d(x_{n-1}, x_{n+1}) \right\} + a_n$$

and hence

$$d(x_n, x_{n+1}) \leqslant \frac{1}{1-q} \cdot a_n + q \cdot d(x_{n-1}, x_n).$$

Proceeding in this manner we get that

$$d(x_n, x_{n+1}) \leqslant \frac{1}{1-q} (a_n + q \cdot a_{n-1} + q^2 a_{n-2} + \cdots + q^{n-1} a_1) + q^n d(x_0, x_1).$$

Then, as the series

$$\sum_{k=1}^{\infty} \left[\frac{1}{1-q} (a_k + q a_{k-1} + \cdots + q^{k-1} a_1) + q^k d(x_0, x_1) \right]$$

is convergent, by routine calculation one can show that $\{x_n\}$ is the Cauchy sequence. Since M is complete there exists $\lim_n T_n x \in M$. Put $Tx = \lim_n T_n x$. Then

$$d(Tx, Ty) = d(\lim_n T_n x, \ \lim_n T_{n+1} y) = \lim_n d(T_n x, T_{n+1} y)$$

$$\leqslant \lim_n \left[q \max \left\{ d x, y); \ d(x, T_n x); \ d(y, T_{n+1} y); \right. \right.$$

$$\left. \left. \frac{1}{2} [d(x, T_{n+1} y) + d(y, T_n x)] \right\} + a_n \right]$$

$$\leqslant q \cdot \max \left\{ d(x, y); \ d(x, Tx); \ d(y, Ty); \ \frac{1}{2} [d(x, Ty) + d(y, Tx)] \right\}.$$

Hence, by Theorem A, it follows that T has a unique fixed point. The proof is complete.

T h e o r e m 4. *Let M be a metric space, and let $\{T_n\}$ be a sequence of mappings which map M into itself. Let $T_0 : M \to M$ be a generalized contraction and let M be T_0-orbitally complete. If each T_n has at least one fixed point u_n and if the sequence $\{T_n\}$ on the subset $I = \{x$: there is some T_k such that $x = T_k x\}$ converges uniformly to T_0, then the sequence $\{u_n\}$ converges to the unique fixed point u_0 of T_0.*

Proof. By Theorem A, T_0 has a unique fixed point u_0. We have

$$d(u_0, u_n) = d(T_0 u_0, T_n u_n) \leqslant d(T_0 u_0, T_0 u_n) + d(T_0 u_n, u_n)$$

$$\leqslant q \cdot \max\left\{d(u_0, u_n);\ d(u_n, T_0 u_n);\ \frac{1}{2}[d(u_0, T_0 u_n) + d(u_n, u_0)]\right\} + d(T_0 u_n, u_n)$$

$$\leqslant q[d(u_0, u_n) + d(u_n, T_0 u_n)] + d(T_0 u_n, u_n)$$

and hence

$$d(u_0, u_n) \leqslant \frac{1+q}{1-q} d(u_n, T_0 u_n).$$

Since $\{T_n\}$ on I converges uniformly to T_0,

$$d(u_n, T_0 u_n) = d(T_n u_n, T_0 u_n) \to 0, \quad n \to \infty$$

and we have that $\lim_n d(u_0, u_n) = 0$ which completes the proof.

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