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SOME REMARKS ON THE COMPLEMENT OF A LINE GRAPH

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Let G be the complement of a line graph. In this paper all triangle-free graphs G are determined.

We shall consider finite, undirected graphs, without loops or multiple lines.

Primarily, we shall repeat or introduce some definitions and basic notions which will be used later.

Line graph of a graph H, denoted by $L(H)$, has as its points the lines of H, with two points being adjacent if the corresponding lines are adjacent in H.

Relative complement of a graph G with respect to a graph U (G must be a spanning subgraph of U), denoted by $C_U G$, is a graph having the same point set as G (or U), with two points being adjacent in $C_U G$ if the corresponding points are adjacent in U but not in G. If U is a complete graph then we get the complement of a graph G, which is denoted by $\bar{G}$.

Next, as usual, let K_n, $K_{m,n}$, C_n, P_n denote complete graph with n points, bicomplete graph with $m+n$ points, cycle of length n and path of length n (n is the number of occurrences of lines in P_n), respectively.

The union of graphs G_1 and G_2 denoted by $G_1 \cup G_2$, is a graph which contains all the points and lines of both graphs G_1 and G_2, and no other points or lines. pG denotes the union of p copies of G. In further text we shall consider also graphs $K_{m,n}-pK_2$ and $K_{m,n}-C_p$. Both of them are spanning subgraphs of a graph $K_{m,n}$ obtained by removal of prescribed number of a certain types of lines. So, if we remove from $K_{m,n}$ p non-adjacent lines (or lines lying on a cycle C_p) we shall get the graph $K_{m,n}-pK_2$ (or $K_{m,n}-C_p$).

And at last let r, d, g denote radius, diameter and girth of the considered graph, respectively. (If these quantities are not determined we shall assume that they are infinite).

In literature there are many results concerning line graphs. This class of graphs is very important and much has been done in finding characterisations of line graphs. It is possible to find some of them for example, in [1]. One of the basic results in the considered area is the result of Beineke [2]. He was able to display exactly all induced subgraphs not occurring in line graphs. The following theorem provides necessary and sufficient conditions.

Theorem 1. *G is a line graph if and only if none of the nine graphs of Fig.* 1 *is an induced subgraph of G.*

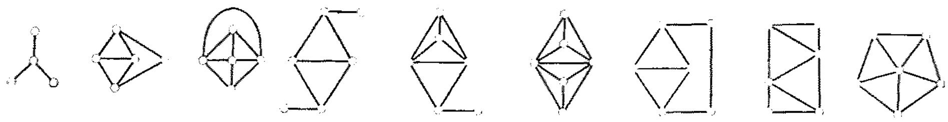

Fig. 1

This theorem can be restated in the following one if we take into consideration a complement of the line graph.

Theorem 2. *G is the complement of a line graph if and only if none of the nine graphs*[1] *of Fig.* 2 *is an induced subgraph of G.*

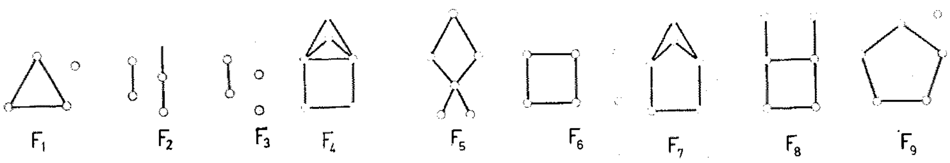

Fig. 2

For the sake of brevity of exposition we shall define for a graph G a property $(\mathcal{P})$:

$(\mathcal{P})$ $\qquad G=\overline{L(H)}$ for some graph H.

Now we can start investigating graphs with property $(\mathcal{P})$. Note, that in [3] all line graphs, having for the complement a line graph too, were determined. The technique used in our paper is similar to that from [3].

We shall begin by proving two lemmas.

Lemma 1. *If G has property* $(\mathcal{P})$, *then G has at most* 3 *components or is totally disconnected. If one of components has at least* 2 *lines then the others are line-free. Graphs in Fig.* 3 *are the only ones having lines in at least two components.*

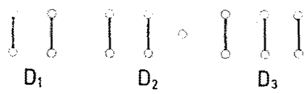

Fig. 3

Proof of Lemma. Suppose that G has more than 3 components so that at least one has a line. Then, G has an induced subgraph F_3, and no property $(\mathcal{P})$. It is not difficult to see that a totally disconnected graph has property $(\mathcal{P})$.

The second part of lemma follows from the fact that a graph having property $(\mathcal{P})$ does not contain F_2 or F_1 as induced subgraphs.

[1] Note that graphs on Fig. 2 are complements of graphs on Fig. 1.

Now, combining the above parts of lemma we can easily get that the graphs in Fig. 3 are the only ones having the lines in at least two components.

Lemma 2. *If G has property* $(\mathcal{P})$, *then G does not contain, as an induced subgraph, cycle* C_n *(or path* P_n*) with* $n \geq 7$ $(n \geq 5)$ *lines.*

Proof of Lemma. The proof immediately follows from the fact that cycle of length 7 (path of length 5) is the shortest one among cycles (paths) containing F_2 as an induced subgraph.

The consequences of Lemma 2 are the following.

Corollary 1. *If G has property* $(\mathcal{P})$ *and at least one cycle, then* $g \leq 6$.

Corollary 2. *If G has property* $(\mathcal{P})$, *each component of G has diameter not greater than* 4.

Theorem 3. *G is a forest having property* $(\mathcal{P})$, *if and only if G is an induced subgraph*[1] *of at least one of the following graphs* T_1, T_2, $S_n = K_{1,n} \cup \overline{K}_2$ $(n = 1, 2, \ldots)$ (see Fig. 4).

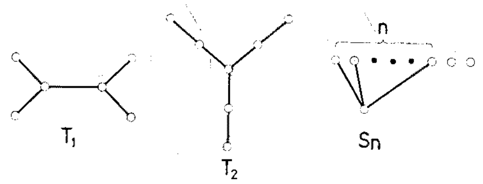

Fig. 4

Proof. Assume G is a forest with property $(\mathcal{P})$. Due to Lemma 1, G has at most 3 components if it contains lines. Otherwise, it is totally disconnected and then it is an induced subgraph of a graph S_n. If at least two components have lines, then G is one of graphs in Fig. 3 and all of them are contained in T_2. Hence, it is sufficient to consider forests having lines in only one component. That component is of course a tree. Due to Corollary 2, $d \leq 4$ for that tree and so $1 \leq r \leq 2$.

We first observe trees having one point in the center.

If $r = 1$, it immediately follows that a tree is a star and so G is, in this case, S_n.

If $r = 2$ for the mentioned tree, there exist at least two points on distance 2 from the center point. The degree of center-point is less than 4 because of the forbidden induced subgraph F_3 (the center-point and some other points, if necessary, are being taken from the tree to get F_3). The points on distance 1 from the center-point are of degree less than 3 because of induced subgraph F_2, which can be found by the similar procedure as above in the case of F_3. If G is disconnected then the degree of center-point of the tree, which is being

[1] In further text the term an induced subgraph of a graph G will include graph G itself.

considered, is 2 and G has only two components because of the forbidden induced subgraph F_3 (it is not difficult to find F_3). Now, it follows that G is an induced subgraph of T_2.

Now, let us take into consideration trees having two adjacent points as the center.

If $r=1$ then G is obviously an induced subgraph of a graph $K_2 \cup \overline{K_2}$ which is itself contained in T_1 or T_2.

If $r=2$, the observed tree contains two adjacent center-points each of them having at least one pendant line and nothing else. The number of pendant lines is less than 3 for each center-point if G is connected or is less than 2 for each center-point if G is disconnected. So we have two possibilities. One is $P_3 \cup K_1$ (more than one isolated point cannot exist due to F_3), contained in graph T_2, and the other is T_1.

This completes the proof.

Theorem 4. *If G is a bipartite graph with $g=6$ and if G has property $(\mathcal{P})$, then G is C_6 or $C_6 \cup K_1$* (see Fig. 5).

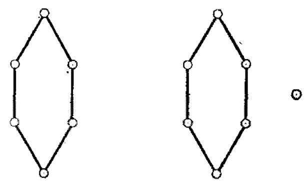

Fig. 5

Proof. G has, by conditions of theorem, a component with at least two lines. By Lemma 1 the other components are isolated points. While a component containing a cycle C_6 has an isolated line and isolated point as an induced subgraph, G can have at most two components. If a component which contains a cycle C_6 has some more points except those on the cycle C_6, then due to $g=6$ there exists a point v adjacent to only one point on the cycle C_6. In that case we can find F_2 as an induced subgraph of that component, so it follows that the graphs of Fig. 5 are all possibilities.

This completes the proof.

Now, we shall describe all bipartite graphs having property $(\mathcal{P})$. (Totally disconnected graphs are not taken into consideration).

Bipartite graphs with 3 components are primarily D_2 and D_3 in Fig. 3. The others contain two isolated points. Suppose, that Q is a component with lines. From Corollary 1 and Theorem 4 it follows that $g<6$ for Q. Further, because of F_6, we have $g \neq 4$. Hence, Q is a tree and by Theorem 3 must be a star.

Except for the graph D_1 in Fig. 3, bipartite graphs with 2 components and property $(\mathcal{P})$ have one isolated point and one non trivial component Q.

Theorem 5. *If G is bipartite graph having two components and property $(\mathcal{P})$, then G is an induced subgraph of one of the following graphs $(K_{m,r} - pK_2) \cup K_1$ $(p \leqslant n \leqslant m;\ p=0, 1, 2, \ldots;\ m, n=1, 2, \ldots)$.*

Proof. Having in view the above restrictions for G from Theorem 5, all the forbidden induced subgraphs[1] of non trivial component are the following (Fig. 6).

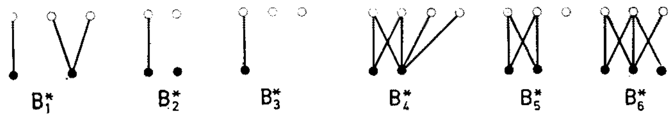

Fig. 6

Now if two components have lines, the only possibility is $K_2 \cup K_2$ which is equal to $K_{2,2}-2K_2$ so that it is sufficient to consider a case when one component Q is non trivial and the other trivial K_1. Because of the forbidden induced subgraph B_3^* and the fact that Q is connected, every point of one colour is adjacent to all points of the opposite colour except for, perhaps, one point and vice versa so that Q is just equal to $K_{m,n}-pK_2$. At the end, we must check whether any of B_i^* $(1 \leqslant i \leqslant 6)$ can be an induced subgraph in Q. The answer is negative since each B_i^* contains a point of one colour non adjacent to two points of opposite colour.

This completes the proof.

Theorem 6. *If G is a connected bipartite graph having property $(\mathcal{P})$, then G is an induced subgraph of at least one of the following graphs*[2]

$$K_{3,3}-C_4,\ K_{3,n}-C_6\ (n=3, 4, \ldots),$$

$$K_{m,n}-pK_2\ (p \leqslant n \leqslant m;\ p=0, 1, 2, \ldots, m,\ n=1, 2, \ldots).$$

Proof. Now, all forbidden induced subgraphs, analoguously as in Theorem 5, are the following (Fig. 7).

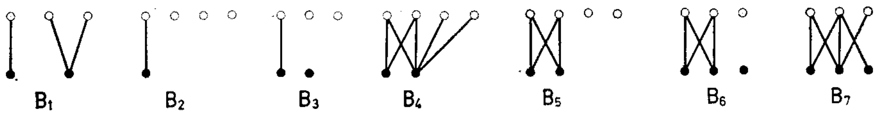

Fig. 7

In this case in order to simplify an analysis it is convenient to take relative complements with respect to the bicomplete graphs. Then the following induced subgraphs are forbidden in CG (Fig. 8).[3]

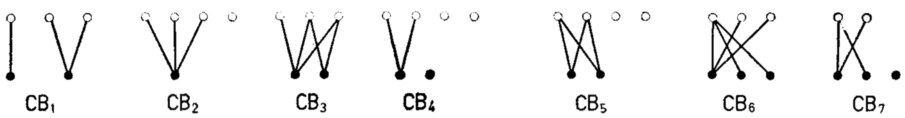

Fig. 8

[1] On Fig. 6 points of different colours are drawn by full and empty circles.

[2] $K_{3,3}-C_4$ is equal to T_1.

[3] CG is an abbreviation for $C_U G$ where U is a bicomplete graph.

Having in view the forbidden induced subgraph CB_2, it can be noticed that the degree of arbitrary point in CG is at most 2, since in CG no point of one colour is adjacent to all points of opposite colour, because of the connectedness of G. Further, because of CB_1, if one component of CG has two lines the others are line-free. Now it is clear that the following cases are possible:

1° CG consists of a cycle of even length and of isolated points.

2° CG consists of a path of a length at least 2 and of isolated points.

3° CG consists of isolated lines and of isolated points.

1° Because of CB_1 the only possible cycles are C_4 or C_6.

Assume at first that CG has a cycle C_4. Then because of CB_5, CG can have at most one isolated point of one colour and it follows from the fact that G is connected, that CG has two isolated points of different colours. It is not difficult to see that in this case G is equal to T_1.

Now, suppose that CG has a cycle C_6. Because of the connectedness of graph G, graph CG must have at least one isolated point. Further because of CB_4 all isolated points must be of the same colour, (otherwise graph CB_4 is obtained by a removal of two points at distance two on the cycle C_6 and some other points if necessary). So graph $K_{3,n}-C_6$ was obtained and it is easy to check that it has property $(\mathcal{P})$.

2° In this case the length of a path is less than 5.

If the length of a path is two then because of connectedness of graph G there must be in CG at least two isolated points of different colours and because of CB_4 all other isolated points must be of the same colour as a point of degree two in the path.

If the length of path is 3 or 4 because of CB_7, all isolated points are of the same colour.

Now, it is easy to see that all graphs G in the case 2° are the induced subgraphs of graph $K_{3,n}-C_6$.

3° This case is quite analoguous with one described in Theorem 5 and it is clear that graph G is now equal to $K_{m,n}-pK_2$.

This completes the proof.

Theorem 7. *If G is not a bipartite graph, if it has no triangles and if it has property $(\mathcal{P})$, then G is an induced subgraph of Pétersen's graph.*

Proof. From Corollary 1 and conditions of the Theorem 7 it is clear that the length of the shortest odd cycle is 5. Because of F_9 all points of G must be either on considered cycle C_5 or adjacent to at least one point of C_5. If one point, which is not on considered cycle C_5, is adjacent to two points of C_5 either triangle or the forbidden induced subgraph F_7 will occur. In the case when the mentioned point is adjacent to more than two points of the considered cycle the triangle will certainly occur. Hence, every point which is not on considered cycle C_5 is adjacent to exactly one point on C_5 and this possibility is allowed. Further, because of F_2, every two non adjacent points which are not on considered cycle C_5 are not adjacent to the same point on C_5. The same is valid for each pair of adjacent points because triangles are not allowed. So, G has at most 10 points.

Suppose that u and v are two points not lying on considered cycle C_5 and let u_c and v_c be the points of C_5 to which u and v are adjacent, respectively. Then the following two possibilities can occur:

1° points u_c and v_c are adjacent,

2° points u_c and v_c are not adjacent.

1° In this case u and v cannot be adjacent because of the induced subgraph F_8.

2° Now, u and v must be adjacent, because of the induced subgraph F_3.

So, on the basis of above conclusions it follows that G is an induced subgraph of Pétersen's graph which has property $(\mathcal{P})$.

This completes the proof.

It is interesting to notice that graphs T_1, T_2 and graphs from Fig. 5 are the induced subgraphs of Pétersen's graph. Also, S_n is an induced subgraph of $K_{3,n}-C_6$.

All above results can be summarized in the following theorem.

Theorem 8. *A graph G with no triangles has a line graph for its complement if and only if G is an induced subgraph of one of the following three graphs:*

(i) $K_1 \cup (K_{m,n}-pK_2)$,

(ii) $K_{3,n}-C_6$,

(iii) *Pétersen's graph* (see Fig. 9).

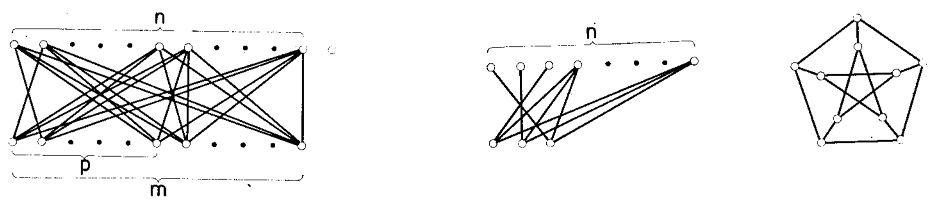

Fig. 9

It can be noticed that by Theorem 8 all graphs H not containing three mutually non adjacent lines by means of $\overline{L(H)}$ are characterized.

Suppose now that G has at least one triangle and property $(\mathcal{P})$. Due to F_1, G is connected and every point of G is adjacent to at least one point of each triangle of G. Hence, we have that $d \leqslant 3$ for G.

Acknowledgement

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