PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série tome 17 (31), 1974, pp, 17–26

ON CERTAIN CLASSES OF COMMUTATORS

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(Received November 16, 1971)

1. Introduction:

Let H be a separable infinite-dimensional (complex) Hilbert space. If C is an operator on H (i.e. a bounded linear transformation of H into itself), then C is said to be a commutator if $C=PQ-QP$ for some operators P and Q on H. The class of commutators for which Q is an irreducible operator will be denoted by M_H. The subclass of M_H for which Q is an irreducible isometry will be denoted by N_H. It is well known that the set of commutators is dense in the set of all operators [2, p. 124]. Then by using the fact that the set of irreducible operators is dense in the set of all operators [7, Theorem 1], one can see that the class M_H is dense in the set of all operators on H. Since every isometry is in the closure of R_1, where R_1 is the set of all operators that have a reducing subspace of dimension 1 [7, Proposition 3], one can prove that each element of N_H is a boundary point of the complement of M_H in the set of all operators on H.

The object of the present paper is to study certain properties of the classes M_H and N_H. Some of these properties are implicit, either in the results, or in their proofs, already known. We have enumerated some of these observations in section I. The central idea behind these observations is that unilateral shift is an irreducible isometry, and an operator unitarily equivalent to an irreducible isometry is an irreducible isometry. In Section 2, sufficient conditions for operators to belong to N_H, $N{H\oplus H\oplus\cdots\oplus H}$, $N{H\oplus H\oplus H\oplus\cdots}$ are obtained. The main result of the paper is Theorem I which is used in almost all the following theorems. In section 3, we prove two theorems on the operators on $H\oplus H$. In the last section, sufficient conditions for operators to belong to M_H are obtained. The methods of proofs are similar to those used by M. David in [3] and [4]. We define a limit point of a normal operator in the usual sense; (see [3]).

1. The following are certain observations regarding the class N_H.

1. Every operator with a large kernel belongs to N_H[6, Prob. 186].
2. Every operator on H is a sum of two elements of N_H. [6, Cor. Prob. 186].
3. N_H is strongly dense in the set of all operators on H. [6, Cor. Prob. 186].

4. $\|1-C\| \geq 1 \ \forall\, C \in N_H$ [6, Prob. 185].

5. Every operator on H can be written as a product of two operators, such that their product in the reverse order belongs to N_H [5, Lemma 2]

6. $O \in \overline{W(C)} \ \forall\, C \in N_H$ [8, Cor. Theorem 1].

7. If $\langle \alpha_n \rangle_{n=1}^{\infty}$ is any bounded sequence of complex numbers containing an addable subsequence, then the diagonal operator A defined on H as $Ax_n = \alpha_n x_n$ where $(x_n)_{n=1}^{\infty}$ is an orthonormal basis for H is an element of N_H [1, Theorem 1].

8. Every weighted unilateral and bilateral shift belongs to N_H. [1, Theorems 2 and 3].

2. Theorem. 1. *Let $C = \langle C_{l,n} \rangle_{l,n=1}^{\infty}$ be an operator matrix defined on $H \oplus H \oplus \cdots$ satisfying*

$$\|C_{l,n}\| \leq \min\left(\frac{1}{L^l}, \frac{1}{L^n}\right) \text{ for all } l \geq 1 \text{ and } n \geq 2$$

Then

$$C \in N_{H \oplus H \oplus H \oplus} \ldots$$

Proof. Let $\cup$ denote the operator matrix whose entries just below the main diagonal equal the identity operator and all other operators are zero. Let

$$A = \begin{bmatrix} -C_{21} & C_{11} & C_{12} & C_{13} & C_{14} \cdots \\ -C_{31} & 0 & C_{11}+C_{22} & C_{12}+C_{23} & C_{13}+C_{24} \cdots \\ -C_{41} & 0 & C_{32} & C_{11}+C_{22}+C_{33} & C_{12}+C_{23}+C_{34} \cdots \\ -C_{51} & 0 & C_{43} & C_{32}+C_{43} & C_{11}+C_{22}+C_{33}+C_{44} \cdots \\ -C_{61} & 0 & C_{52} & C_{42}+{}_{53} & C_{32}+C_{43}+C_{54} \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

One can prove that $C = A\cup - \cup A$. The proof of the theorem will be completed if we are able to show that A is a bounded operator. Now $A = X + Y + Z$, where

$$X = \begin{bmatrix} -C_{21} & 0 & 0 \cdots \\ -C_{31} & 0 & 0 \cdots \\ -C_{41} & 0 & 0 \cdots \\ -C_{51} & 0 & 0 \cdots \\ \vdots & \vdots & \vdots \end{bmatrix} \qquad Y = \begin{bmatrix} 0 & C_{11} & 0 & 0 \cdots \\ 0 & 0 & C_{11} & 0 \cdots \\ 0 & 0 & 0 & C_{11} \cdots \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

$$Z = \begin{bmatrix} 0 & 0 & C_{12} & C_{13} & C_{14} \cdots \\ 0 & 0 & C_{22} & C_{12}+C_{23} & C_{13}+C_{24} \\ 0 & 0 & C_{32} & C_{22}+C_{33} & C_{12}+C_{23}+C_{33} \cdots \\ 0 & 0 & C_{42} & C_{32}+C_{43} & C_{22}+C_{33}+C_{44} \cdots \\ 0 & 0 & C_{52} & C_{42}+C_{53} & C_{32}+C_{43}+C_{54} \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

Since C_{11} is a bounded operator, Y is bounded. Let $y = \langle y_n \rangle_{n=1}^{\infty}$ be a vector in $H \oplus H \oplus H \oplus \ldots$ Then $Xy = y'$ where $y' = \langle -C_{j1}\, y_1 D \rangle_{j=2}^{\infty}$. Let $y'' = (y_1, 0, 0 \ldots$

be another vector of $H\oplus H\oplus H\oplus\dots$ Then $Cy''=y'''$ where $y'''=\langle C_{j1}\,y_1\rangle_{j=1}^{\infty}$ Therefore.

$$\|Xy\|^2=\sum_{j=2}^{\infty}\|C_{j1}\,y_1\|^2\leqslant\sum_{j=1}\|C_{j1}\,y_1\|^2.$$

But

$$\|Cy''\|^2=\sum_{j=1}^{\infty}\|C_{j1}y_1\|^2\leqslant K\|y''\|^2=K\|y_1\|^2$$

for some constant K. Therefore

$$\|Xy\|^2\leqslant K\|y_1\|^2\leqslant K\sum_{j=1}^{\infty}\|y_j\|=K\|y\|^2\;\forall\,y$$

which proves X to be a bounded operator.
Next using the hypothesis

$$\|C_{l,\,n}\|\leqslant\min\left(\frac{1}{L^l},\ \frac{1}{L^n}\right)\ \text{for all } l\geqslant 1 \text{ and } n\geqslant 2$$

We see that the matrix Z is dominated entrywise by the following Toeplitz matrix

$$\begin{bmatrix}
1 & 1 & 1 & \frac{1}{L^2} & \frac{1}{L^3} & \frac{1}{L^4}\cdots\\
\frac{1}{L^4} & 1 & 1 & 1 & \frac{1}{L^2} & \frac{1}{L^3}\cdots\\
\frac{1}{L^4} & \frac{1}{L^3} & 1 & 1 & 1 & \frac{1}{L^2}\cdots\\
\frac{1}{L^5} & \frac{1}{L^4} & \frac{1}{L^3} & 1 & 1 & 1\cdots\\
\frac{1}{L^6} & \frac{1}{L^5} & \frac{1}{L^4} & \frac{1}{L^3} & 1 & 1\cdots\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{bmatrix}$$

Hence Z is a bounded operator [1, Lemma 3.3].

We shall need the following results:

Lemma 1. *Let $C_1, C_2, \dots, C_p$ be a finite set of compact operators on H. Let K be an infinite dimensional subspace of H. Also let $\varepsilon_2, \varepsilon_3, \dots \varepsilon_n$, $2\leqslant n\leqslant\infty$ be a set of positive numbers. Then K can be written in the form*

$$K=\sum_{i=1}^{n}\oplus K_i$$

where K_i are mutually orthogonal infinite-dimensional subspaces of K and $\|C_jx\|\leqslant\leqslant\varepsilon_i\|x\|$ for each $x\neq 0$ in K_i, $i=1, 2, \dots$, *n and for $j=1, 2, \dots, p$.* (See David [3, Lemma 1]).

Lemma 2. *Let C be a normal operator on H having 0 as a limit point of its spectrum. Let $\varepsilon_2 > \varepsilon_3 > \varepsilon_4 > \cdots$ be a sequence of positive numbers converging to 0, then there exists a sequence $\langle M_i \rangle_{i=1}^{\infty}$ of mutually orthogonal infinite-dimensional subspaces of H, such that*

(a) $H = \sum_{i=1}^{\infty} \oplus M_i$

(b) M_i *reduces* C $\forall i$.

(c) $\|C/M_i\| < \varepsilon_i$ for $i = 2, 3, \ldots$

(See Brown, Halmos and Pearcy [1, Theorem 5]).

Theorem 2. *Let*

$$S = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ A_{n-1,1} & A_{n-2,2} & \cdots & A_{n-1,n} \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{bmatrix}$$

be an operator on $H \oplus H \oplus \cdots \oplus H$ (H taken n times in the sum), *where each* A_{ij} *is an operator on* H. *If*

(*i*) $A_{1n}, A_{2n}, \ldots, A_{n-1,n}$ *are compact operators and*

(*ii*) $A_{nn} = N + G$ *where* N *is a normal operator having* 0 *as a limit point of its spectrum and* G *is a compact operator, then*

$$S \in N_{H \oplus H \oplus \cdots \oplus H}.$$

Proof. By lemma 2, let

$$H = \sum_{i=1}^{\infty} \oplus M_i$$

be the representation of H corresponding to the normal operator N and to the sequence $\varepsilon_j = \frac{1}{2nL^{j+1}}$. By lemma 1, we have for $i = 2, 3, \ldots$

$$M_i = L_i \oplus N_i$$

where L_i and N_i are orthogonal infinite-dimensional subspaces of M_i such that

$$\|A_{jn} x\| < \frac{1}{2n L^{i+1}} \|x\|, \ \|A_{jn}^* x\| < \frac{1}{2n L^{i+1}} \|x\|, \ j = 1, 2, \ldots, n-1 \ \text{and}$$

$$\|Gx\| < \frac{1}{2n L^{i+1}} \|x\|, \ \|G^* x\| < \frac{1}{2n L^{i+1}} \|x\|, \ \forall x \neq 0 \ \text{in} \ L_i, \ i = 2, 3, \ldots$$

Define

$$H_2 = M_1 \oplus \sum_{i=2}^{\infty} \oplus N_i \ \text{and} \ H_i = L_{i-1}, \ i = 3, 4, \ldots$$

Then

$$H=\sum_{i=2}^{\infty}\oplus H_i$$

and for each $x\in H_i$ $x\neq 0$ and $i=3, 4,\ldots$

$$\|A_{jn}x\|<\frac{1}{2nL^i}\|x\|,\ \|A_{jn}^*x\|<\frac{1}{2nL^i}\|x\|,\ j=1, 2,\ldots, n-1 \text{ and}$$

$$\|Gx\|<\frac{1}{2nL^i}\|x\|,\ \|G^*x\|<\frac{1}{2nL^i}\|x\|,\ \|Nx\|<\frac{1}{2nL^i}\|x\|.$$

Let P_i be the projection on H_i $i=2, 3,\ldots$ and let

$$Q_1=\begin{bmatrix} I & 0 & 0 & \cdots & 0 & 0\\ 0 & I & 0 & \cdots & 0 & 0\\ \vdots & & & & & \\ 0 & 0 & 0 & \cdots & I & 0\\ 0 & 0 & 0 & \cdots & 0 & P_2\end{bmatrix},\quad Q_i=\begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0\\ 0 & 0 & 0 & \cdots & 0 & 0\\ \vdots & & & & & \\ 0 & 0 & 0 & \cdots & 0 & 0\\ 0 & 0 & 0 & \cdots & 0 & P_{i+1}\end{bmatrix}\quad i=2, 3,\ldots$$

Then

$$H\oplus H\oplus\cdots\oplus H=\sum_{k=1}^{\infty}\oplus Q_K(H\oplus H\oplus\cdots\oplus H).$$

where

$$Q_1(H\oplus H\oplus\cdots\oplus H)=H\oplus H\oplus\cdots\oplus H\oplus H_2 \text{ and}$$
$$Q_i(H\oplus H\oplus\cdots\oplus H)=0\oplus 0\oplus\cdots\oplus H_{i+1}\qquad i=2, 3,\ldots$$

Let

$$U_1=\begin{bmatrix} I & I & \cdots & I & V_2\\ 0 & 0 & \cdots & 0 & 0\\ \vdots & & & & \\ 0 & 0 & \cdots & 0 & 0\end{bmatrix}\quad\text{and}\quad U_i=\begin{bmatrix} 0 & 0 & \cdots & 0 & V_{i+1}\\ 0 & 0 & \cdots & 0 & 0\\ \vdots & & & & \\ 0 & 0 & \cdots & 0 & 0\end{bmatrix}\quad i=2, 3\ldots$$

where I is the identity operator on H and each V_i is an operator on H mapping H_i isometrically on H and annihilating $H\ominus H_i$. Then U_1 maps $Q_1(H\oplus H\oplus\cdots\oplus H)$ isometrically onto $H\oplus 0\oplus\cdots\oplus 0$ and annihilates $(H\oplus H\oplus\cdots\oplus H)\ominus Q_1(H\oplus H\oplus\cdots\oplus H)$ and U_i maps $Q_i(H\oplus H\oplus\cdots\oplus H)$ isometrically onto $H\oplus 0\oplus 0\oplus\cdots\oplus 0$ and annihilates $(H\oplus H\oplus\cdots\oplus H)\ominus Q_i H\oplus H\oplus\cdots\oplus H)$ and therefore by [1, Lemma 3.1] S is unitarily equivalent to an operator on $H\oplus H\oplus H\oplus\cdots$ where $C_{ij}=U_i S U_j^*$ for all i and j. We have simple calculations for $j\geqslant 2$

$$C_{1j}=(A_{1n}V_{j+1}^*+A_{2n}V_{j+1}^*+\cdots+A_{n-1,n}V_{j+1}^*+V_2NV_{j+1}^*+$$
$$+V_2GV_{j+1}^*)\oplus 0\oplus 0\oplus\cdots+0,$$

and for $i\geqslant 2$, $j\geqslant 2$

$$C_{ij}=(V_{i+1}NV_{j+1}^*+V_{i+j}GV_{j+1}^*)\oplus 0\oplus 0\oplus\cdots\oplus 0.$$

Let $(x, 0, 0 \ldots . 0)$ be a vector in $H \oplus 0 \oplus 0 \oplus \cdots + 0$. Then for $j \geqslant 2$

$$\|C_{1j}(x, 0, 0 \ldots 0)\| = \|(A_{1n} V_{j+1}^* + A_{2n} V_{j+1}^* + \cdots + A_{n-1n} V_{j+1}^* + V_2 G V_{j+1}^* + + V_j N V_{j+1}^*) x\| < \frac{1}{2n L^{j+1}} \|x\| + \frac{1}{2n L^{j+1}} \|x\| + \cdots + \frac{1}{2n L^{j+1}} \|x\| + + \frac{1}{2n L^{j+1}} \|x\| < \frac{1}{L^{j+1}} \|x\| < \frac{1}{L^j} \|x\|$$

Therefore

$$\|C_{1j}\| < \frac{1}{L^j}\ j = 2, 3, \ldots$$

For $i \geqslant 2$ and $j \geqslant 2$

$$\|C_{ij}(x, 0, 0 \cdots 0)\| = \|(V_{i+1} N V_{j+1}^* + V_{i+1} G V_{j+1}^*) x\| < \frac{1}{2n L^{j+1}} \|x\| + + \frac{1}{2n L^{j+1}} \|x\| < \frac{1}{L^j} \|x\|$$

and

$$\|C_{ij}^*(x, 0 \cdots 0)\| = \|(V_{j+1} N^* V_{i+1}^* + V_{j+1} G^* V_{i+1}^*) x\| < \frac{1}{2n L^{i+1}} \|x\| + \frac{1}{2n L^{i+1}} \|x\| < \frac{1}{L^i} \|x\|$$

Therefore

$$\|C_{ij}\| = \|C_{ij}^*\| \leqslant \min\left(\frac{1}{L^i}, \frac{1}{L^j}\right)$$

for all $i \geqslant 1$ and $j \geqslant 2$ and hence by Theorem 1, $C \in N_{H \oplus H \oplus H \oplus \cdots}$. Since S is unitarily equivalent to C, therefore the result follows.

From Theorem 2 we get

Theorem 3. *If N is a normal operator having 0 as a limit point of its spectrum and G is a compact operator then $N + G \in N_H$.*

3. Theorem 4. *If P and Q are operators on H such that $P + Q \in M_H$, then $P \oplus Q \in M_{H \oplus H}$.*

Proof. Let $P + Q = AB - BA$ where B is an irreducible operator. Then one can observe that the commutator of

$$\begin{bmatrix} 0 & A \\ AB\text{-}P & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & I \\ B & 0 \end{bmatrix}$$

is $P \oplus Q$. Also

$$\begin{bmatrix} 0 & I \\ B & 0 \end{bmatrix}$$

is an irreducible operator on $H \oplus H$.

The proof shows the following

Cor. If $P+Q\in N_H$, then $P\oplus Q\in N_{H\oplus H}$.

One can see that the converse of Theorem 4 is not true by considering the operators P and Q defined as follows:

$$Px_n = 2x_n \;\forall\, n$$

$$Qx_n = -x_n \;\forall\, n$$

where $\langle x_n \rangle_{n=1}^{\infty}$ is an orthonormal basis for H. Then $P\oplus Q$ is a self adjoint operator having limit points 2 and -1 and therefore by Theorem 6, which we shall prove in the next section, $P\oplus Q\in M_{H\oplus H}$. But $P+Q=I$.

Theorem 5. *If $P, Q\in N_H$, then every operator of the form*

$$\mathcal{Z}=\begin{bmatrix} P & R \\ S & Q \end{bmatrix}$$

for arbitrary R and S belongs to $M_{H\oplus H}$.

Proof. Let $P=A_1B_1-B_1A_1$ and $Q=A_2B_2, \;-B_2A_2$ where A_1 and A_2 are irreducible operators. Choose real numbers b_1 and b_2 such that $\mathrm{sp}(B_1+b_1 I)\cap \cap\,\mathrm{sp}(B_2+b_2 I)=\Phi$. Therefore by a theorem of Rosenblum ([9]). The equations

$$X(B_1+b_1 I)-(B_2+b_2 I)X=S \qquad \ldots \quad (i)$$

and

$$X(B_2+b_2 I)-(B_1+b_1 I)X=R \qquad \ldots \quad (ii)$$

have a solution. We can also write $P=A_1(B_1+b_1 I)-(B_1+b_1 I)A_1$ and $Q= =A_2(B_2+b_2 I)-(B_2+b_2 I)A_2$. ($B_1+b_1 I$ and $B_2+b_2 I$ are also irreducible operators.) Let the solutions of (*i*) and (*ii*) be respectively denoted by C and D. Then one can observe that the commutator of

$$\begin{bmatrix} A_1 & D \\ C & A_2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} B_1+b_1 I & 0 \\ 0 & B_2+b_2 I \end{bmatrix}$$

is the given operator.

It is possible that an operator

$$\begin{bmatrix} P & R \\ S & Q \end{bmatrix}$$

may belong to $N_{H\oplus H}$ while none of P, Q, R, S is in N_H. For example

$$\begin{bmatrix} I & I \\ I & -I \end{bmatrix}$$

is such an operator.

If, $P, Q\in N_H$, then operator of the type $\mathcal{Z}$ will belong to $N_{H\oplus H}$ for arbitrary R and S is not known to the author.

4. In this section we prove the following

Theorem 6. *If N is a normal operator having 0 in the convex hull of the set of limit points of* $\mathrm{Sp}(N)$ *and G is a compact operator, then $N+G\in M_H$.*

For the proof we require the following lemma

Lemma 3. *Let N_j $j=1, 2, 3, 4, 5, 6$ be six normal operators and let* 0 *be a limit point of* $\mathrm{Sp}(N_j)\,\forall_j$. *Then there exist six unitary operators U_j such that*

$$S=\begin{bmatrix} A & B & E_1+G_1 \\ C & D & E_2+G_3 \\ E & F & E_3+G_3 \end{bmatrix}$$

is in $NH\oplus H\oplus H$ for any six operators A, B, C, D, E, F, three compact operators G_i and three operators $E_1=\sum_{j=1}^{6} a_{ij}\, U_j^*\, N_j\, U$, *where a_{ij} are complex numbers.*

Proof of the Lemma. Using Lemma 2, H can be written in the form

$$H=\sum_{k=1}^{\infty} \oplus M_K$$

such that

(*i*) M_K are mutually orthogonal infinite-dimensional subspaces of H.

(*ii*) There exist six unitary operators U_j on H with $U_1=I$ such that M_K are reducing subspaces for $U_j^* N_j U_j$ for $j=1, 2, 3, 4, 5, 6$

(*iii*) $\|U_j^* N\ U_j|M_K\| < \dfrac{1}{72\,L^{k+1}}$ for $j=1, 2, 3, 4, 5, 6$ and for $k=2, 3, 4\ \ldots$

Let $C>\sup\limits_{i,j}|a_{ij}|$ and let $E_i'=\dfrac{1}{c}E_i$. Since each M_K is reduced by E , therefore each M_K is reduced by E'. Thus 1) M_K reduces E' and for $i=1, 2, 3$.

2) $\|E'|M_K\| = \left\|\dfrac{1}{c}\sum_{j=1}^{6} a_{ij}U^*N_jU_j\,\middle|\, M_K\right\| < \dfrac{1}{c}\times\dfrac{6\,\mathrm{Sup}\,|a_{ij}|}{72\,L^{k+1}}=\dfrac{1}{12\,L^{k+1}}$

Let $G'=\dfrac{1}{c}G_i$. By lemma 1 let

$$M_K=L_K\oplus N_K \qquad K=2, 3, \ldots$$

where L_K and N_K are orthogonal infinite dimensional subspaces of M_K such that

$$\|G'x\| < \frac{1}{12\,L^{k+1}}\|x\| \text{ and } \|G_i'^*x\| < \frac{1}{12\,L^{k+1}}\|x\|$$

for each $x\neq 0$ in L_K and for $i=1, 2, 3, \ldots$

Write

$$H_2=M_1\oplus\sum_{k=2}^{\infty}\oplus N_K \text{ and } H_i=L_{i-1} \text{ for } i=3, 4, \ldots$$

Then

$$H=\sum_{i=2}^{\infty}\oplus H_i$$

such that

$$\|G' x\| < \frac{1}{12\,L^k}\|x\|,\ \|G_i'^* x\| < \frac{1}{12\,L^k}\|x\|,\ \|E_i' x\| < \frac{1}{12\,L^k}\|x\| \text{ and}$$

$$E_i'^* x\| < \frac{1}{12\,L^k}\|x\|$$

for $i=1, 2, 3$ and for each $x\neq 0$ in H_K $k=3, 4, \ldots$
Let

$$Q_1 = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & P_2 \end{bmatrix} \quad \text{and } Q_i = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & P_{i+1} \end{bmatrix}$$

where P_i, $i=2, 3, \ldots$ are projections on H_i, $i=2, 3, \ldots$
Then

$$H\oplus H\oplus H = \sum_{i=1}^{\infty} \oplus Q_i\,(H\oplus H\oplus H)$$

where

$$Q_i\,(H\oplus H\oplus H) = H\oplus H\oplus H_2$$

and

$$Q_2\,(H\oplus H\oplus H) = 0\oplus 0\oplus H_{i+1} \quad i=2, 3, \ldots$$

Let

$$U_1 = \begin{bmatrix} I & I & V_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad U_i = \begin{bmatrix} 0 & 0 & V_{i+1} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad i=2, 3, \ldots$$

where each V_i is an operator on H mapping H_i isometrically on H and annihilating $H\ominus H_i$. Similarly, therefore, as in Theorem 2, the operator

$$S' = \frac{1}{c}S = \begin{bmatrix} A' & B' & E_1'+G_1' \\ C' & D' & E_2'+G_2' \\ E' & F' & E_3'+G_3' \end{bmatrix}$$

where $A' = \dfrac{1}{c}A$ etc. is unitarily equivalent to an operator matrix $\langle C_{ij}\rangle_{i,j=1}^{\infty}$ such that $C_{ij} = U_i\cdot S' U_j^* \,\forall\ i$ and j. We have for $j\geqslant 2$

$$C_{1j} = (E_1' V_{j+1}^* + G_1' V_{j+1} E_2' V_{j+1}^* + G_2' V_{j+1}^* + V_1 E_3' V_{j+1}^* + V_1 G_3' V_{j+1}^*)\oplus 0\oplus 0$$

and for $i\geqslant 2$, $j\geqslant 2$

$$C_{ij} = (V_{i+1} E_3' V_{j+1}^* + V_{i+1} G_3' V_{j+1}^*)\oplus 0\oplus 0$$

One can prove similarly that

$$\|C_{l,n}\| \leqslant \min\left(\frac{1}{L^l}, \frac{1}{L^n}\right)$$

for all $l\geqslant 1$ and $n\geqslant 2$, and therefore $S' \in N_{H\oplus H\oplus H}$. Hence $S = CS' \in N_{H\oplus H\oplus H}$.

Proof of Theorem 6. If N is a normal operator having 0 in the convex hull of the set of limit points of $\mathrm{Sp}(N)$ and G is a compact operator, then $N+G$ is unitarily equivalent to an operator of the form

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

where A and D are operators of the form S in Lemma 3 [4, Theorem 1]. Hence $A, D \in N_H$. Hence $N+G \in M_H$ by Theorem 5.

Let N be an operator on H. Consider the following

(a) there exists an infinite orthonormal sequence $\langle e_j \rangle$ in H such that

$$\left| \sum_{j=1}^{n} (N e_j, e_j) \right| \text{ is bounded } \forall n.$$

(b) N is not of the form $N_1 \oplus N_2$ where N_1 has finite dimensional domain and N_2 satisfies

$$\inf_{\|x\|=1} |(N_2 x, x)| > 0$$

(c) 0 is in the convex hull of the set of limit points of $\mathrm{Sp}(N)$.

We observe that (b) is a consequence of (a) for arbitrary operators [4] and (c) is a consequence of (b) for normal operators [3]. Hence we conclude the following

Theorem 7. *If a normal operator N satisfies any of* (a), (b), *or* (c) *then $N+G \in M_H$ for any compact operator G.*

The author expresses his sincere gratitude to Dr. B. S. Yadav for his helpful suggestions during the preparation of this paper.

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