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MONOTONIC MAPPINGS ON ORDERED SETS, A CLASS OF INEQUALITIES WITH FINITE DIFFERENCES AND FIXED POINTS

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0. Introduction.

In this paper, we shall consider monotonic mappings of ordered sets and, in connexion with it, a class of inequalities with finite differences. Further, using obtained results, we will prove two theorems on fixed points. We have been inspirated by an idea of S. B. Prešić [1].

1. Let $(0, \leqslant)$ be a set ordered by the order relation $\leqslant$, and let $\circ$ be a binary operation on 0 with inverse operation $\alpha(\circ)$ and satisfying the conditions

$$(A)\qquad (a)\ (x, y, z \in 0)\quad x \leqslant z \circ y \Rightarrow x\,\alpha(\circ)\,y \leqslant z,$$

$$(b)\ (x, y, z \in 0)\quad x \circ (z \circ y) = z \circ (x \circ y),$$

$$(c)\ (x, y, z \in 0)\quad x \leqslant y \Rightarrow z \circ x \leqslant z \circ y.$$

Definition 1. *Let $(0, \leqslant)$ be an ordered set. We say that the mapping $f: 0^k \to 0$ $(k \in N)$ is increasing, resp. decreasing, if*

$$(x_i, y_i \in 0 \wedge x_i \leqslant y_i\ (i = 1, 2, \ldots, k)) \Rightarrow f(x_1, \ldots, x_k) \leqslant f(y_1, \ldots, y_k),$$

resp.

$$(x_i, y_i \in 0 \wedge x_i \leqslant y_i\ (i = 1, \ldots, k)) \Rightarrow f(x_1, \ldots, x_k) \geqslant f(y_1, \ldots, y_k).$$

We shall also consider semi-homogeneoys functions, defined as follovs.

Definition 2. *The mapping $f: 0^k \to 0$ $(k \in N)$ is called semi-homogenous on the ordered set $(0, \leqslant)$ if*

$$(B)\qquad f(\lambda \circ x_1, \ldots, \lambda \circ x_k) \leqslant \lambda \circ f(x_1, \ldots, x_k)\ (\lambda, f(x_1, \ldots, x_k) \in 0)$$

and homogeneous if

$$(C)\qquad f(\lambda \circ x_1, \ldots, \lambda \circ x_k) = \lambda \circ f(x_1, \ldots, x_k)\ (\lambda, f(x_1, \ldots, x_k) \in 0),$$

where „$\circ$“ is a given operation on O.

It is clear that $(B) \Rightarrow (C)$.

In our considerations we shall use the following property of an ordered set $(0, \leqslant)$:

$(\mathcal{P})$ *For any two elements a and b of O, the set $\{a, b\}$ has an upper bound.* Evidently, in this case every finite subset of 0 has an upper bound, too. We denote by $a = \mathcal{M}(S)$ the fact that a is an upper bound of S.

Our results

Proposition 1. *Let $(0, \leqslant)$ be a set ordered by the order relation $\leqslant$ and having the property $(\mathcal{P})$, and let for the relation $\leqslant$ and the operation „$\circ$" a 0 be satisfied the conditions (A). Suppose that the mapping $f: 0^k \to 0$, with a fixed natural number k, is increasing and semi-homogenous, and that two sequences x_n and X_n of elements of 0 satisfy the conditions*

$$(1) \qquad x_{n+k} \leqslant f(a_1(n) \circ x_n, \ldots, a_k(n) \circ x_{n+k-1}),$$

$$f(a_1(n) \circ X_n, \ldots, a_k(n) \circ X_{n+k-1}) \leqslant X_{n+k}$$

where $a_1(n), \ldots, a_k(n) \in 0$. Then there exists an element $L \in 0$ such that

$$(2) \qquad x_n \leqslant L \circ X_n \; (n \in N).$$

Proof. Let us put

$$L = \mathcal{M}\{x_1 \alpha(\circ) X_1, \ldots, x_k \alpha(\circ) X_k\}.$$

We shall prove that then (2) holds. We apply the mathematical induction. Since

$$x_i \leqslant L \circ X_i \Leftrightarrow x_i \alpha(\circ) X_i \leqslant L \; (i = 1, \ldots, k),$$

(2) holds for $n = 1, \ldots, k$. Let us suppose that (2) holds for $n, n+1, \ldots, n+k-1$, i.e. that

$$(3) \qquad x_{n+i-1} \leqslant L \circ X_{n+i-1} \qquad (i = 1, \ldots, k).$$

Then we have, by (1) and (3),

$$\begin{aligned} x_{n+k} &\leqslant f(a_1(n) \circ x_n, \ldots, a_k(n) \circ x_{n+k-1}) \\ &\leqslant f(a_1(n) \circ (L \circ X_n), \ldots, a_k(n) \circ (L \circ X_{n+k-1})) \\ &\leqslant f(L \circ (a_1(n) \circ X_n), \ldots, L \circ (a_k(n) \circ X_{n+k-1})) \\ &\leqslant L \circ f(a_1(n) \circ X_n, \ldots, a_k(n) \circ X_{n+k-1}) \\ &\leqslant L \circ X_{n+k}. \end{aligned}$$

So (2) holds for every $n \in N$.

Remark. If we omit the supposition (*b*) characterising the ordered set $(0, \preccurlyeq)$ we also can prove proposition I. The proof is similar, but the relations (1) should be replaced by

$$x_{n+k} \preccurlyeq f(x_n, \ldots, x_{n+k-1})$$
$$X_{n+k} \succcurlyeq f(X_n, \ldots, X_{n+k-1}), \qquad (n \in N).$$

Proposition 2. 1° *Let $f: R^k \to R$ $(k \in N)$ be and monotonically increasing and semihomogenous mapping a let the sequence (x_n) of real numbers satisfy the condition*

$$x_{n+k} \leqslant f(a_1 x_n, \ldots, a_k x_{n+k-1}) \qquad (n = 1,2, \ldots);$$

k fixed natural number where $a_1, \ldots, a_k$ are real constants. Then there exist positive numbers L and θ such that

$$x_n \leqq L \cdot \theta^n \qquad (n = 1,2, \ldots). \tag{4}$$

2° *Especially, if f is a continuous mapping and $|f(|a_1|, \ldots, |a_k|)| < 1$, then there exists a $\theta \in (0,1)$ with property* (4).

Proof. It will be done in several steps. We begin by the following

Lemma. *Let the mapping $f: R^k \to R$ $(k \in N)$ be increasing and semi homogeneous. Then for given $a_1, a_2, \ldots, a_k \in R$ there exists a positive solution of the inequality*

$$f(a_1, a_2 x, \ldots, a_k x^{k-1}) \leqslant x^k \qquad (k \in N).$$

Proof. The mapping $g: R^+ \to R^+$ (R^+ the set of all positive numbers) defined by

$$g(x) \overset{\text{def}}{=} f\left(\frac{|a_1|}{x^{k-1}}, \ldots, \frac{|a_{k-1}|}{x^1}, |a_k|\right) \tag{5}$$

is decreasing. By our suppositions, there exists a $\theta \in R$ such that $g(\theta) < \theta$, i.e.

$$f(a_1, a_2 \theta, \ldots, a_k \theta^{k-1}) \leqq f(|a_1|, |a_2| \theta, \ldots, |a_k| \theta^{k-1}) \leqslant \theta^k \tag{5'}$$

q.e.d.

We continue with the proof of our proposition. According to Lemma, there exists a $\theta \in R_+$ such that

$$\theta^k \geqslant f(a_1, a_2 \theta, \ldots, a_k \theta^{k-1}).$$

Then the sequence $X_n = \theta^n$ of positive numbers satisfies the condition

$$X_{n+k} \geqslant f(a_1 X_n, \ldots, a_k X_{n+k-1}) \qquad (n \in N),$$

so that we conclude, by the proposition 1 (for $0 = R$ and substituting $\preccurlyeq$ to $\leqslant$), that there exists $L \in 0$ such that

$$x_n \leqslant L X_n (n = 1,2, \ldots),$$

i.e.

$$x_n \leqq L \theta^n (n = 1,2, \ldots)$$

which was to be proved.

Proof. of 2°. Since the mapping f is continuous, for the mapping $g: R^+ \to R^+$ defined by (5) there exists $\theta \in (0,1)$ with the property $g(\theta) \leqslant \theta$, i.e.

$$f\left(\frac{|a_1|}{\theta^k}, \ldots, \frac{|a_k|}{\theta}\right) \leqq 1,$$

which implies (5′).

Remark. When the semihomogeneity in Proposition 2 is substituted by

$$f(\lambda x_1, \ldots, \lambda x_k) \leqq \lambda f(x_1, \ldots, x_k), \ \lambda \in [\alpha, \infty) \subset R$$

then also are valid the conditions 1 and 2 of the Proposition 2 beside other suppositions. The proof is similar, only that the element $\mathscr{L} \in R$ is

$$\mathscr{L} = \max\left(\alpha, \frac{x_1}{\theta}, \ldots, \frac{x_k}{\theta^k}\right), \qquad (\theta > 0)$$

Similarly in Proposition 1 can this condition be restricted. By the previous proof, our assertion holds.

1.2. We shall expose some applications of previous propositions.

Example 1. Lemma 1 of *S. Prešić* given in [1] is a special case of our Proposition 1. It is sufficient to put $0 = R$, $\preccurlyeq$ to substitute $\leqq$ to $\circ$, to substitute the ordinary multiplication to $\cdot$, and to applicate Proposition 1 with $f: R^k \to R$ defined by

$$f(x_1, \ldots, x_k) = x_1 + \ldots + x_k.$$

In the same manner one can deduce Lemma 2 from [1] from our Proposition 2. We cite this lemma:

Lemma. (*Prešić* [1], p. 76.) *Let x_n be a sequence of non negative numbers satisfying the condition*

$$x_{n+k} \leqq a_1 x_n + \cdots + a_k x_{n+k-1} \qquad (n = 1, 2, \ldots)$$

$a_1, a_2, \ldots, a_k$ being non negative constansts. Then there exist positive numbers L and θ such that (4) *holds.*

Let us cite another mere concrete example.

Example 2. Let $\{x_n\}$ and $\{X_n\}$ be sequences of non negative numbers satisfying

$$x_{n+k} \leqq \frac{a x_n + b}{c x_n + d} + \cdots + \frac{a x_{n+k-1} + b}{c x_{n+k-1} + d},$$

$$X_{n+k} \geqq \frac{a X_n + b}{c X_n + d} + \cdots + \frac{a X_{n+k-1} + b}{c X_{n+k-1} + d}$$

$$(n = 1, 2, \ldots;\ a, b, c, d, > 0;\ ad - cb > 0;\ X_1, \ldots, X_k > 0).$$

Then we have, with a positive constant L,

$$x_n \leqq L X_n \, (n = 1, 2, \ldots).$$

P r o o f. The mapping $f: R^{+} \to R^{+}$ given by

$$f(x) = \frac{ax+b}{cx+d} \qquad (a, b, c, d>0; \quad ad-bc>0)$$

satisfies all conditions of Propositition 1. The application of this theorem gives our assertion.

A consequence of our results is also

C o r o l l a r y 2. (*Dj. Kurepa*, [2], p. 103.) *If we have in a metric space*

$$\rho[x_{n+1}, x_n] \leqq \lambda_1 \rho[x_n, x_{n-1}] + \cdots + \lambda_k \rho[x_{n-k+1}, x_{n-k}] \ (\lambda_i \geqq 0),$$

where $\lambda_1 + \cdots + \lambda_k \in [0,1)$, *then the sequence* $\{x_n\}$ *is fundamental.*

P r o o f. This assertion follows immediately from the part 2° of Proposition 2, since the sequence $(\rho[x_n, x_{n+1}])$ has the properties of the sequence $\{x_n\}$ mentioned in this proposition.

We give another example illustrating the application of Proposition 2:

E x a m p l e 3. Let $x_n \geqq 0$ and

$$x_{n+4} \leqq \frac{ax_n+b}{cx_n+d} + \ln(1+\alpha_1 x_{n+1} + \alpha_2 x_{n+2} + \alpha_3 x_{n+3})$$

$$(ad-bc>0; \ a, b, c, d>0, \alpha_i \geqq 0; \ n \in N)$$

Then:

1. there exists $\mathscr{L} \in R_+$ such that $x_n \leqq \mathscr{L}\theta^n (n \in N, \theta>0)$,
2. if

$$\frac{a+b}{c+d} + \ln(1+\alpha_1+\alpha_2+\alpha_3) < 1,$$

then $x_n \to 0 \ (n \to \infty)$.

E x a m p l e 4. Let $T: X \to X$ where X is a complete metric space and ρ its metric, where for all $x, y \in X$, $a_i \in R_+$

$$\text{(D)} \quad \rho[Tx, Ty] \leqq \max\left(a_1 \rho[x, y], \ a_2 \rho[x, Tx], \ a_3 \rho[y, Ty], \ \frac{a_4}{2}\rho[x, Ty], \ a_5 \rho[y, Tx]\right),$$

and $\max(a_1, \ldots, a_k) \in [0, 1)$, then there exists a unique fixed point $\xi \in X$ of the mapping $T: X \to X$.

E x a m p l e 5. The above assertion is valid (from the Example 4.) even when the condition (D) is substituted by

$$\rho[Tx, Ty] \leqq \ln\left(1 + a_1 \rho[x, y] + a_2 \rho[x, Tx] + a_3 \rho[y, Ty] + \frac{a_4}{2}\rho[x, Ty] + a_5 \rho[y, Tx]\right)$$

and then $\ln(1+a_1+\cdots+a_5) \in [0, 1)$.

Remark. Proposition 1 can be extended to pairs of inequalities with finite differences having the form

$$x_n \leqslant f_n(a_0(n), a_1(n)\circ x_1 \ldots, a_{n-1}(n)\circ x_{n-1}),$$

$$X_n \geqslant f_n(a_0(n). a_1(n)\circ X_1, \ldots, a_{n-1}(n)\circ X_{n-1}),$$

or to pairs of systems of inequalities of this type. An example of results of this kind is the following

Proposition 3. *Let* $(O, \leqslant)$ *be an ordered set,* $f_n : O^{2n-1} \to O$ $(n \in N)$ *a monotonically increasing and semi homogeneous mapping, and* x_n, y_n, X_n, Y_n *sequences satisfying*

$$x_n \leqslant f_n(a_0(n), a_1(n)\circ x_1, \ldots, a_{n-1}(n)\circ x_{n-1}, b_1(n)\circ y_1, \ldots, b_{n-1}(n)\circ y_{n-1})$$

$$y_n \leqslant f_n(c_0(n), c_1(n)\circ x_1, \ldots, c_{n-1}(n)\circ x_{n-1}, d_1(n)\circ y_1, \ldots, d_{n-1}(n)\circ y_{n-1})$$

$$X_n \geqslant f_n(a_0(n), a_1(n)\circ X_1, \ldots, a_{n-1}(n)\circ X_{n-1}, b_1(n)\circ Y_1, \ldots, b_{n-1}(n)\circ Y_{n-1})$$

$$Y_n \geqslant f_n(c_0(n), c_1(n)\circ X_1, \ldots, c_{n-1}(n)\circ X_{n-1}, d_1(n)\circ Y_1, \ldots, d_{n-1}(n)\circ Y_{n-1})$$

where $a_i(n), b_i(n), c_i(n), d_i(n) \in O$ $(i = 1,2, \ldots, n-1)$. *Then there exist elements* $L_1, L_2 \in O$ *such that*

$$x_n \leqslant L_1 \circ X_n, \quad y_n \leqslant L_2 \circ Y_n \quad (n = 1,2, \ldots).$$

Here $\circ$ *is an operation on* O *with property* (A).

The proof is similar to the proof of Proposition 1.

Corollary. Using the mapping $f : R^k \to R$ defined by

$$f_n(x_1, \ldots, x_k) = \sum_{i=1}^{2n-1} x_i \ (x_i \geqq 0),$$

as in the case of Corollary 2, we obtain the corresponding lemma of S. Prešić ([1], p. 77.).

2. Let (X, ρ) be a complete metric space, k a given natural number and the mapping $f : R^k \to R$ be continuous, increasing and semi homogenous; let further T be a mapping of X^k to X and let satisfy the condition

$$\rho[Tx, Ty] \leqq |f(a_1 \rho[u_1, v_1], \ldots, a_k \rho[u_k, v_k])|; \tag{6}$$

$$x = (u_1, \ldots. u_k), \ y = (v_1, \ldots, v_k); \ u_i, v_i \in X (i = 1, \ldots, k),$$

where $a_1, \ldots, a_k$ are positive numbers such that

$$|f(a_1, \ldots, a_k)| < 1. \tag{7}$$

Remark. Specifying the mapping f and the dimension k, one can with help of (6) and (7) obtain sufficient conditions which lead to the operators of contraction given by Banach [6], Kannan [4], Lj. Ćirić [3], M. Tasković [5], S. Prešić [1], Đ. Kurepa [2]. Especially, if the mapping $f: R^k \to R$ is defined by

$$f(x_1, \ldots, x_k) = x_1 + \cdots + x_k,$$

we obtain the operators of S. Prešić introduced in [1].

The Paper [5] contains a further result in this direction.

Theorem 1. *Suppose that the mapping $T: X^k \to X$, with fixed $k \in N$, has the properties* (6) *and* (7). *Then*:

(I) *There exists a unique fixed point $\xi \in X$ of the mapping $\mathcal{G}(x) \overset{\text{def}}{=} T(x, \ldots, x)$.*

(II) *ξ is the limit of the sequence $\{x_n\}$ satisfying*

$$x_{n+k} = T(x_n, \ldots, x_{n+k-1}) \qquad (n = 1,2,\ldots) \tag{8}$$

independently of initial values.

(III) *The rapidity of convergence of the sequence $\{x_n\}$ to the point ξ is evaluated by*

$$\rho[x_{n+k}, \xi] \leq L \frac{\theta^n}{1-\theta}, \quad \theta \in (0,1), \quad (n = 1,2,\ldots),$$

where $L = \max_{1 \leq i \leq k} \left\{ \frac{\rho[x_i, x_{i+1}]}{\theta^i} \right\}$.

Proof. Applying Proposition 2 to the sequence $(\rho[x_n, x_{n+1}])$ one gets

$$\rho[x_{n+k}, x_{n+k+1}] \leq L\theta^n (n = 1,2,\ldots); \ \theta \in (0,1),$$

and consequently

$$\rho[x_{n+k}, x_{n+k+p}] \leq L \frac{\theta^n}{1-\theta} \qquad (n, p = 1,2,\ldots), \tag{9}$$

which implies that (8) is a Cauchy sequence, and consequently converges in X. The inequality of the fixed point ξ, follows from the inequality

$$\rho[\xi, \xi^*] \leq \rho[\xi, \xi^*] \, |f(a_1 \ldots, a_k)|,$$

and the relation in (III) is obtained from (9) for $p \to \infty$.

Remark. The previous considerations have various applications, and, especially can be usefully applied in further investigations in the domain of problems of fixed points and Banach's mappings. (See the paper [5]).

We quote still some sufficient conditions that can be obtainked on the base of the condition (6), by choosing adequately the mapping $f: R \to R^k$.

Example 6. A sufficient condition corresponds to f given by

$$f(x_1, \ldots, x_k) = \max\{x_1, \ldots, x_k\} \text{ or } f(x_1, \ldots, x_k) = \min\{x_1, \ldots, x_k\},$$

such that the condition (6) can be substituted by the condition

$$\rho[Tx, Ty] \leq \max\{a, \rho[u_1, v_1], \ldots, a_k \rho[u_k, v_1]\}.$$

Example 7. $f: R^2 \to R\,(k=2)$;

$$f(x, y) = \frac{x^2}{x-y}\,(x > 2\,y).$$

This example was usefully employed in the paper [5], where was effectively showed that this mapping determines a new class of sufficient conditions for the existence of fixed points.

It is possible to apply the preceding theorem to the resolution of algebraic, differential, integral and other equations. These applications are not yet sufficiently examined.

Theorem 2. *Let $T: X^k \to X\,(k \in N)$ where X be a complete metric space and ρ its metric. If for all $x, y \in X^k$ there exist nonnegative numbers α, β, γ* and *$q_i\,(i=1,2,\dots,k)$ such that*

$$\text{(10)} \qquad \sup_{x,\, y \in X^k} \Big\{\alpha + \beta + 2\,\gamma + \sum_{(i=1,\dots,k)} q_i\Big\} = \lambda < 1$$

and

$$\rho[Tx, Ty] \leqq \alpha(x, y)\,\rho[u_k, Tx] + \beta(x, y)\,\rho[v_k, Ty] + \gamma(x, y)\,\rho[u_k, Ty] +$$

$$+ \gamma(x, y)\,\rho[v_k, Tx] + \sum_{(i=1,\dots,k)} q_i\,\rho[u_i, v_i]$$

where $x = (u_1, \dots, u_k)$, $y = (v_1, \dots, v_k)$, then the assertions I, II, and III in the theorem 1 are valid.

Proof Follows from *Proposition 2*, with $f(t_1, \dots, t_k) = \sum_{(i=1,2,\dots,k)} \lambda_i x_i$ to the sequence $(\rho[x_n, x_{n+1}])$; we obtain, according to (4):

$$\rho[x_{n+k}, x_{n+k+1}) = \rho[T(x_n, \dots, x_{n+k-1}), T(x_{n+1}, \dots, x_{n+k})] \leqq$$

$$\leqq \alpha(\overline{x}_n, \overline{x}_{n+1})\,\rho[x_{n+k-1}, x_{n+k}] + \beta(\overline{x}_n, \overline{x}_{n+1})\,\rho[x_{n+k}, x_{n+k+1}] + \gamma(\overline{x}_n, \overline{x}_{n+1})$$

$$\rho[x_{n+k-1}, x_{n+k+1}] + \sum_{(i=1,2,\dots,k)} q_i(\overline{x}_n, \overline{x}_{n+1})\,\rho[x_{n+i-i}, x_{n+i}]$$

where $\overline{x}_n \overset{\text{def}}{=\!=} (x_n, \dots, x_{n+k-1})$. Therefore

$$(1-\beta-\gamma)\,\rho[x_{n+k}, x_{n+k+1}] \leqq q_1\,\rho[x_n, x_{n+1}] + \cdots + q_{k-1}\,\rho[x_{n+k-2}, x_{n+k-1}] +$$

$$+ (\alpha + \gamma + q_k)\,\rho[x_{n+k-1}, x_{n+k}]$$

and

$$\rho[x_n, x_{n+1}] \leqq \theta^n \max_{(i=1,2,\dots,k)} \left\{\frac{\rho[x_i, x_{i+1}]}{\theta^i}\right\} (n \in N,\ \theta \in (0,1))$$

Hence, for $n, s \in N$

$$(11) \qquad \rho[x_n, x_{n+s}] \leqq \sum_{(j=1,\ldots,s)} \rho[x_{n+j-1}, x_{n+j}] \leqq$$

$$\leqq \max_{(i=1,\ldots,k)} \left\{\frac{\rho[x_i, x_{i+1}]}{\theta^i}\right\} \sum_{(j-1,\ldots,k)} \theta^{n+j-1} \leqq \frac{\theta^n}{1-\theta} \max_{i=1,\ldots,k} \left\{\frac{\rho[x_i, x_{i+1}]}{\theta^i}\right\}$$

which implies that (x_n) is a Cauchy's sequence. Hence, the metric space X being complete, there exists

$$\xi = \lim_{n\to\infty} x_n$$

Let us prove that ζ is a fixed point of T (in the sense precised above). We get, according to our hypothesis on T,

$$\rho[x_{n+k}, T(\xi,\ldots,\xi)] = \rho[T(x_n,\ldots,x_{n+k-1}), T(\xi,\ldots,\xi)] \leqq$$

$$\leqq \alpha\rho[x_{n+k-1}, x_{n+k}] + \beta\rho[\zeta, T(\xi,\ldots,\xi)] + \gamma\rho[x_{n+k-1}, T(\xi.\ldots,\xi)] +$$

$$+ \gamma\rho[\xi, x_{n+k}] + \sum_{(i=1,\ldots k)} q_i\,\rho[x_{n+i-1}, \xi]$$

and further, similarly as in the above considerations,

$$(1-\beta-\gamma)\,\rho[x_{n+k}, T(\xi,\ldots,\xi)] \leqq \alpha\rho[x_{n+k-1}, x_{n+k}] + \beta\rho[\xi, x_{n+k}] +$$

$$+ \gamma\rho[x_{n+k-1}, x_{n+k}] + \gamma\rho[\xi, x_{n+k}] \sum_{(i=1,\ldots,k)} q_i\,\rho[x_{n+i-1}, \xi].$$

Hence,

$$\xi = \lim_{n\to\infty} x_{n+k} = T(\xi,\ldots,\xi) \stackrel{\text{def}}{=} \mathcal{P}(\xi)$$

which was to be proved.

We will prove, finally that the fixed point ξ is unique. Let us suppose that $\xi^* \neq \xi$ is a fixed point too. Then

$$\rho[\xi, \xi^*] = \rho[T(\xi,\ldots,\xi), T(\xi^*,\ldots\xi, ^*)] \leqq \alpha\rho[\xi, T(\xi,\ldots,\xi)] +$$

$$+ \beta\rho[\xi^*, T(\xi^*,\ldots,\xi^*)] + \gamma\rho[\xi, T(\xi^*,\ldots,\xi^*)] + \gamma\rho[\xi^*, T(\xi,\ldots,\xi)] +$$

$$+ \sum_{(i=1,\ldots,k)} q_i\,\rho[\xi, \xi^*]$$

and consequently

$$(1-2\gamma)\,\rho[\xi, \xi^*] \leqq \Big(\sum_{i=1,\ldots,k} q_i\Big)\,\rho[\xi, \xi^*]$$

i.e. since $\rho[\xi, \xi^*] > 0$

$$1 \leqq 2\gamma + \sum q_i$$

which contradicts (10). This contradiction proves our assertion. Making $S \to \infty$ in (11), one gets (III). The proof is complete.

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REFERENCES

[1] Prešić S. *Sur une classe d'inéquations aux differences finies et sur la convergence de certaines suites*, Publ. Inst. Math. 5 (19), 1965, pp. 75-78.

[2] Kurepa Đ., *Some fixed point theorems*, Math. Balkanica 2 (1972) pp. 102-108.

[3] Ćirić Lj., *Generalized centractions and fixed point theorem*, Publ. Inst Math. N. S. 12 (26) (1971)

[4] Kannan R., *Some results on fixed points — II*, Amer. Math. Monthly, 76 (1969)

[5] Tasković M. *Generalized f_λ — contractions and Fixed point theorems*, Math. Balkanica 3 (1973)

[6[Banach S., *Sur Les opérations dans les ensembles abstraits et leur applications aux équations intégrales* Fund. Math., 3 (1922)