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ON MULTIVALUED QUOTIENT MAPPINGS

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0. Introduction

This paper is concerned with properties of multi-valued quotient mappings, which extend the concept of quotient mappings for single-valued functions to multivalued functions.

Section 2. contains some equivalent conditions for a multi-valued mapping to be an us-quotient (ls-quotient) mapping (Theorem 2.2.).

The main results in Section 3. are: (*a*) a relationship between multi-valued quotient mappings and almost single-valued mappings (Theorem 3.2.) and (*b*) in analogy with the case of single-valued mappings, a Y-connected quotient multi-valued mapping preserves local connectedness (Theorem 3.10.).

Some other results are: a multivalued quotient mapping maps a connected space onto a connected or a discrete space; and a multi-valued Y-closed quotient mapping maps a locally connected compact space X onto a locally connected space Y.

1. Preliminaries

For any sets X and Y, $F:X\to Y$ is a multi-valued mapping provided that, for each $x\in X$, $F(x)$ is a nonempty subset of Y.

Definition 1.1 Let $F:X\to Y$ be a multi-valued mapping. Then

(1) $F(A)=\cup\{F(x):x\in X\}$ for each $A\subset X$,

(2) $F'(B)=\{x\in X:F(x)\cap B\neq\varnothing\}$ for each $B\subset Y$,

(3) $\overset{\circ}{F}(A)=C\circ F\circ C(A)=\{y\in Y:F'(y)\subset A\}$ for each $A\subset X$ (C-denotes complement),

(4) $\overset{\circ}{F'}(B)=C\circ F'\circ C(B)=\{x\in X:F(x)\subset B\}$ for each $B\subset Y$.

If $F:X\to Y$ is a multi-valued mapping, then $F(x)$ need not be a closed set as required, for example, in [2], [4], [5], [6]. For this reason we adopt the following convention. Let P be a property of sets. Then a multi-valued mapping $F:X\to Y$ is called $Y-P$ $(X-P)$ if and only if $F(x)$ $(F'(y))$ has property P for each $x\in X$ (for each $y\in Y$). Properties we are going to use in this paper are closed, compact and connected.

Let $F: X \to Y$ be a multi-valued mapping. Denote by $P(Y)$ the nonempty subsets of Y. Then F induces a single-valued function $\tilde{F}: X \to P(Y)$ by setting $\tilde{F}(x) = F(x)$ for all $x \in X$, and $\hat{F}: P(X) \to P(Y)$ by setting $\hat{F}(A) = F(A)$ for all $A \in P(X)$.

Define (following M. Marjanović [3]) some topologies on nonempty subsets of a topological space.

Definition 1.2. Let $(X, \mathcal{T}_X)$ be a topological space and $\mathcal{A}$ a family of subsets of X containing the family $\mathcal{T}_X$ ($\mathcal{T}_X$ — family of all open subsets of X). Then

(1) $P_\varkappa(X, \mathcal{A})$ is the topological space having for ist elements all nonempty subsets of X and for the open subbase of its topology the collection of all

$$\langle U \rangle, \quad U \in \mathcal{A} (\langle U \rangle = \{A \subset X : A \subset U\}),$$

(2) $P_\lambda(X, \mathcal{A})$ is the topological space having for its elements all nonempty subsets of X and for the open subbase of its topology the collections of all

$$\rangle U \langle, \quad U \in \mathcal{A} (\rangle U \langle = \{A \subset X : A \cap U \neq \varnothing\}),$$

(3) $P_\psi(X, \mathcal{A})$ is the topological space having for its elements all nonempty subsets of X and for the open subbase of its topology the collections of all

$$\langle U \rangle \text{ and } \rangle V \langle, \quad U \in \mathcal{A}, \quad V \in \mathcal{A}.$$

Remark. If $\mathcal{A} = \mathcal{T}_X$ we denote $P_\varkappa(X, \mathcal{T}_X)$, $P_\lambda(X, \mathcal{T}_X)$ and $P_\psi(X, \mathcal{T}_X)$ by $P_\varkappa(X)$, $P_\lambda(X)$ and $P_\psi(X)$ respectively.

If

$$\exp(X) = \{E \subset X : E \text{ is closed and nonempy}\},$$

then $\varkappa(X, \mathcal{A})$, $\lambda(X, \mathcal{A})$, $\psi(X, \mathcal{A})$ are the topological spaces on closed subsets defined by (1), (2), (3) respectively.

For the continuity of the multi-valued mapping the following definition is used.

Definition 1.3. Let $X \to Y$ be a multi-valued mapping. Then

(1) F is *upper semi-continuous* (*u.s.c.*) provided that $F'(B)$ is closed for each closed $B \subset Y$,

(2) F is *lower semi-continuous* (*l.s.c.*) provided that $F'(V)$ is open for each open $V \subset Y$,

(3) F is *continuous* provided that F is an upper semi-continuous and a lower semi-continuous mapping.

The proof of Theorem 1.4. is omitted (see, for example [6]).

Theorem 1.4. *Let X and Y be topological spaces and $F: X \to Y$ be a multi-valued mapping. Then the following assertions are equivalent:*

(1) *F is an u.s.c. mapping (l.sc. mapping) (continuous mapping),*

(2) *$\tilde{F}: X \to P_\varkappa(Y)$ is continuous ($\tilde{F}: X \to P_\lambda(Y)$ is continuous) ($\tilde{F}: X \to P_\psi(Y)$ is continuous),*

(3) *$\hat{F}: P_\varkappa(X) \to P_\varkappa(Y)$ is continuous ($\hat{F}: P_\lambda(X) \to P_\lambda(Y)$ is continuous) ($\hat{F}: P_\psi(X) \to P_\psi(Y)$ is continuous).*

2. Equivalence of some conditions

The following definition is given by C. J. R. Borges ([1] p.p. 457.).

Definition 2.1. Let X and Y be topological spaces and $F:X\to Y$ an onto multi-valued mapping. Then F is said to be an *us-quotient* (*ls-quotient*) mapping provided that a subset B of Y is closed (open) if and only if $F'(B)$ is a closed subset of $X(F'(B)$ is an open subset of $X)$. F is said to be a *quotient* mapping whenever F is both an us-quotient mapping and a ls-quotient mapping.

If $F:X\to Y$ is such that $F(x)=Y$ for all $x\in X$, then F is continuous but is not quotient. It is easily seen that this mapping F is quotient if and only if Y is a discrete space.

As a direct consequence of the definition given for us-quotient and ls-quotient mapping, the composition of multi-valued mappings which are us-quotient (ls-quotient) is us-quotient (ls-quotient). (If $F_1:X\to Y$ and $F_2:Y\to Z$, then $F_2\circ F_1:X\to Z$ is defined by $F_2\circ F_1(x)=F_2(F_1(x))$).

The following result gives in a way a characterisation of us-quotient, ls-quotient and quotient multi-valued mappings.

Theorem 2.2 *Let $F:X\to Y$ be an onto multi-valued mapping. Then the following conditions are equivalent.*

(1) *F is an us-quotient (ls-quotient) mapping,*

(2) *If $B\subset Y$, then B is an open subset of Y (B is a closed subset of Y) if and only if $\overset{\circ}{F}'(B)$ is an open (closed) subset of X,*

(3) *If $B\subset Y$, then B is open in Y if and only if $\tilde{F}^{-1}(\langle B\rangle)$ is open in X ($\tilde{F}^{-1}(\rangle B\langle)$ is open in X,*

(4) *For any topological space Z and mapping $G:Y\to Z$ we have G is an u.s.c. mapping (l.s.c. mapping) if and only if $G\circ F:X\to Z$ is u.s.c. (l.s.c.).*

Proof. The equivalence of (1), (2) and (3) follows from Definitions 1.1., Definition 2.1. and Theorem 1.4.

We now concentrate attention to the case of us-quotient mapping.

(1) $\Rightarrow$ (4). Let $F:X\to Y$ be an us-quotient mapping and $G:Y\to Z$ be a mapping where Z an arbitrary topological space. If G u.s.c., then $G\circ F$ is also u.s.c. Let, now $G\circ F$ be an u.s.c. mapping and B be a closed subset of Z. Then $(G\circ F)'(B)=F'(G'(B))$ is a closed subset of X and, since F is us-quotient, we have that $G'(B)$ is closed in Y and G is u.s.c.

(4) $\Rightarrow$ (3). Suppose (3) is not satisfied. Then (*a*) there exists an open nonempy subset $V\subset Y$ such that $\tilde{F}^{-1}(\langle V\rangle)$ is not open, i.e. F is not u.s.c.; or

(*b*) there exists a nonopen subset $B\subset Y$ such that $\tilde{F}^{-1}(\langle B\rangle)$ is an open subset of X.

Case (*a*). If F satisfies the condition (4) then F is u.s.c. (take Z to be Y and G the identity function). Hence, in case (*a*), F does not satisfy the condition (4).

Case (b). Let $\mathcal{B}=\{B\subset Y:\tilde{F}^{-1}(\langle B\rangle)$ is open in $X\}$ and let $Z=P_{\varkappa}(Y,\mathcal{B})$. Then the mapping $F:X\to P_{\varkappa}(Y,\mathcal{B})$ is continuous. Let $G:P_{\varkappa}(Y)\to Z$ be such that $G(B)=B$ for all $B\subset Y$. Then the mapping G is not continuous. The mapping $G\circ\tilde{F}:X\to Z$ is continuous and F does not satisfy the condition (4).

The proof concerning ls-quotient mapping is simalar where only one takes $P_{\lambda}(Y,\mathcal{B})$ instead of $P_{\varkappa}(Y,\mathcal{B})$.

Now we have.

Corollary 2.3. *Let $F:X\to Y$ be an onto multi-valued mapping. Then the following conditions are equivalent.*

(1) *F is an us-quotient mapping (ls-quotient mapping).*

(2) *The mapping $\tilde{F}:X\to P_{\varkappa}(Y,\mathcal{B})$ (the mapping $\tilde{F}:X\to P_{\lambda}(Y,\mathcal{B})$) is continuous if and only if $\mathcal{B}=\mathcal{T}_X$.*

3. Relationship between quotient and single-valued mapping

In [5] Ponomarev has introduced the concept of almost singe-valued mapping (see also [4]). In the following definition the mapping F need not be a Y-closed as required by Ponomarev.

Definition 3.1. Let $F:X\to Y$ be an onto multi-valued mapping. Then F is said to be an almost single-valued mapping provided that $\overset{\circ}{F'}(V)\neq\varnothing$ for each open nonempty subset V of Y.

The following Theorem gives a relationship between multi-valued quotient mappings and the almost single-valued mappings.

Theorem 3.2. *Let $F:X\to Y$ be an us-quotient (ls-quotient) mapping, X T_1-space and Y dense in itself T_1-space. Then F is almost single-valued mapping.*

Proof. Suppose F is us-quotient and F si not almost single-valued. Then there exists an open nonempty subset $V\subset Y$ but such that $\overset{\circ}{F'}(V)=\varnothing$ or equivalently $F'(CV)=X$. Then for every $y_0\in V$ we have that $C\{y_0\}\supset CV$ and $F'(C\{y_0\})=X$. Since F is us-quotient and $F'(C\{y_0\})$ is closed, then $C\{y_0\}$ is closed and $\{y_0\}$ is open, that contradicts the supposition that Y is dense in itself T_1-space.

Let now F be a ls-quotient mapping and F is not almost single-valued. Then there exists an open nonempty subset $V\subset Y$ but such that $\overset{\circ}{F'}(V)=\varnothing$. Since F is ls-quotient and $\varnothing$ closed, we have that V is closed, (see Theorem 2.2. (2)). Let $y_0\in V$. Then we have that $F'(\{y_0\}\cup CV)=X$ and $\{y_0\}\cup CV$ is open, which contradicts that Y is dense in itself T_1-space.

Let $f:X\to Y$ be an onto single-valued function. It is well known that if f is an open or closed continuous function, then f is a quotient function. For multi-valued mappings it is not true. The following theorem gives a necessary condition that an almost single-valued function is a quotient multi--valued mapping.

Theorem 3 3. *Let $F:X\to Y$ be an onto, open, closed and X-compact almost single-valued continuous mapping, where, X and Y are T_1-space. Then F is a quotient mapping.*

Proof. Let $F'(B)$ be a closed subset of X and let $y_0\in\overline{B}$. Since F is X-compact and almost single-valued mapping, then there exists a $x_0\in X$ such that $F(x_0)=y_0$ (see [5] Lemma 2. p.p. 534.). Since F is u.s.c. and open, then we have $F'(\overline{B})=\overline{F'(\overline{B})}=\overline{F'(B)}$ (F open $\Rightarrow$ F' l.s.c. $\Rightarrow$ $F'(\overline{B})\subseteq\overline{F'(B)}$; F u.s.c. $\Rightarrow$ F' closed $F'(\overline{B})=\overline{F'(\overline{B})}$ and we have that $\overline{F'(\overline{B})}\subseteq F'(B)$ and $F'(\overline{B})=\overline{F'(B)}$). To the hypothesis $y_0\in\overline{B}$. So $x_0\in F'(\overline{B})=F'(B)$. Hence $y_0=F(x)\subset B$, and so B is closed in Y and F is us-quotient.

Let $F'(B)$ be an open subset of X and let $y_0\in B$. Then $F'(y_0)\subset F(B)$ and, since F is X-compact, then there exists an open subset U_0 in X such that $F'(y_0)\subset U_0\subset F'(B)$. To the hypothesis F is a closed multi-valued mapping and we have that $\overset{\circ}{F}(U_0)$ is an open subset of Y and $y_0\in\overset{\circ}{F}(U_0)$. If $y_1\in\overset{\circ}{F}(U_0)$, then $F'(y_1)\subset U_0\subset F'(B)$. If $y_1\notin B$ and if $y_1=F(x_1)$, then $F(x_1)\cap B=\varnothing$ and $x_1\notin F'(B)$ and this contradicts the relation $x_1\in F'(y_1)\subset F'(B)$. Hence $\overset{\circ}{F}(U_0)\subset B$, and so B is open and F is a ls-quotient mapping.

Lemma 3.4. *Let $F:X\to Y$ be a multivalued quotient mapping, where X and Y are T_1-space. If y is not an isolated point of Y, then there exists $x\in X$ such that $F(x)=y$, i.e. $\overset{\circ}{F}'(y)\neq\varnothing$.*

Proof. If $\overset{\circ}{F}'(y)=\varnothing$, then y is an open subset of Y which contradicts that y is not an isolated point of Y.

Lemma 3.5. *Let $F:X\to Y$ be an onto multi-valued continuous mapping, where X is a connected space. Then $F(x)$ intersect every closed and open subset of Y for each $x\in X$.*

Proof. Let $B\subset Y$ be an open and closed subset. Then $F'(B)$ is an open closed subset of X. Since X is a connected space, then $F'(B)=X$ and we have $F(x)\cap B\neq 0$ for each $x\in X$.

From Lemma 3.5. we have the following:

Theorem 3.6. (Theorem 1. in [6] p.p. 209). *Let $F:X\to Y$ be a continuous multi-valued mapping, and X a connected space. If there is at least one point $x\in X$ such that $F(x)$ is a connected set, then Y is connected.*

From Lemma 3.4. and Theorem 3.6. we have the following.

Corollary 3.7. *Let $F:X\to Y$ be a multi-valued quotient mapping from the connected space X onto a T_1-space Y. Then Y is connected or discrete.*

Proposition 3.8 Let $F:X\to Y$ be a multi-valued quotient mapping and let X be a T_1-space and X has the finite number of components and Y T_1-space. Then the set of all isolated points of Y is a closed and open set.

P r o o f. Denote by Y_0 the set of all isolated points of Y and by Y_1 the set of all nonisolated points of Y. The set Y_0 is open. If $Y_0 = \varnothing$, or $Y_0 = Y$, or Y_0 is finite, then Y_0 is closed.

Let $Y_0 \neq \varnothing$ and $Y_1 \neq \varnothing$ and let $C_1, C_2, \dots, C_n$ be all components of the space X.

If $C_i \cap F'(Y_0) = \varnothing$ for $i = 1, 2, \dots, n$ then $\overset{\circ}{F'}(Y_0) = \varnothing$ and since F is quotient, we have that Y_0 is closed and open.

If $x_k \in \overset{\circ}{F'}(Y_0) \cap C_k \neq 0$ then $F(x_k) \subset Y_0$ and from Lemma 3.5. $F(x_k)$ intersects every closed and open subset of $F(C_k)$ and we have that $F(C_k) \subset Y_0$.

Let $C_1, C_2, \dots, C_m$ be such that $\overset{\circ}{F}(Y_0) \cap C_i \neq 0$ for $i = 1, 2, \dots, m$. Then

$$\overset{\circ}{F'}(Y_0) = C_1 \cup C_2 \cup \cdots \cup C_m$$

and Y_0 is closed and open.

It is known that if f is an open, closed or quotient single valued mapping then f maps a locally connected space onto a locally connected space Consequently, if $f: X \to Y$ is a continuous single valued mapping from a compact locally connected T_2-space X onto a T_2-space Y, then Y is also locally connected. For multi-valued mappings this is not generally true.

Ponomarev shows the following result ([5] p. p. 534. Theorem 2.).

T h e o r e m (Ponomarev). *Let F be a continuous Y-closed almost single valued mapping from a compact locally connected space X onto a T_2-space Y. Then Y is locally connected.*

Now we can apply the above results to obtain the following theorem.

T h e o r e m 3.9. *Let be a multivalued Y-closed quotient mapping from the locally connected compact space X onto a T_2-space Y. Then Y is locally connected.*

P r o o f. Let $C_1, C_2, \dots, C_n$ be all components of X (since X is a compact and locally connected space, then X has the finite number of components).

Denote by Y_0 the set of all isolated points of Y and by Y_1 the set of all nonisolated points of Y. Since the space Y is locally connected at each isolated point in Y, it is necessary to show that Y_1 is locally connected.

Since Y_1 is an open and closed set (see Proposition 3.8.) and F is a quotient multi-valued mapping, then the set $\overset{\circ}{F'}(Y_1) \subset X$ is open, closed and non empty. Let $\overset{\circ}{F'}(Y_1) = C_1 \cup C_2 \cup \cdots \cup C_n = X_1$, $(m < n)$ and let $F_1: X_1 \to X_1$ be the restriction of F to X_1 $(F_1 = F / X_1)$.

Since F_1 is a continuous almost single-valued mapping and X_1 a compact locally connected space, we get that Y_1 is a locally connected space, by above Theorem (Ponomarev).

T h e o r e m 3.10. *Let $F: X \to Y$ be a multi-valued Y-connected quotient mapping, X locally connected T_1-space and Y T_1-space. Then Y is a locally connected space.*

Proof. Let $y_0 \in Y$ be a nonisolated point and V an arbitrary neighbourhood of y_0. If C is the component of y_0 in V, then we must show that the component C is an open subset of Y. By Lemma 3.4. there exists a point $x_0 \in X$ such that $F(x_0)=y_0$. The set $\overset{\circ}{F}'(V_0)$ is an open and it is a neighbourhood of x_0, and $\overset{\circ}{F}'(C) \subset \overset{\circ}{F}'(V_0)$.

Let $x \in \overset{\circ}{F}'(C)$. Since X is a locally connected space, then the component C_x of x in $\overset{\circ}{F}'(V)$ is an open subset in X. The mapping F is Y-connected and we have that $F(C_x)$ is a connected subset of Y, and $F(C_x) \cap C \neq \varnothing$. Since $y_0 \in F(C_x) \cup C \subset \overset{\circ}{F}'(V_0)$, $F(C_x) \cup C$ is connected and C is the component of y_0 in Y, then we have that $F(C_x) \subset C$.

From $F(C_x) \subset C$ follows that $C_x \subset \overset{\circ}{F}'(C)$ and $\overset{\circ}{F}'(C)$ is an open set in X. Since F is a quotient mapping and $\overset{\circ}{F}'(C)$ is an open set, then C is an open subset of Y. Hence Y is a locally connected space.

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