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ON UNSTEADY THREE-DIMENSIONAL BOUNDARY LAYER

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1. Introduction

The recent decade has seen an everincreasing amount of attention to problems of unsteady fluid mechanics. A class of such unsteady flows are unsteady boundary layer flows in two or three-dimensions. The boundary layer equations with appropriate boundary conditions are solved in various situations subjected to the assumptions of the separableness of space variables from time. Such method of treatement greatly simplify the governing equations and subsequently a solution of interest can be exactly or approximatively obtained by analytical or numerical methods. The resulting solution has some interest and application in predicting the structures of the unsteady boundary layers.

In the present paper an elegant mathematical treatement of the three-dimensional incompressible, unsteady, laminar boundary layer flow with the main-stream velocities of the forms $U(x)\,\Omega(t)$ and $V\,\Omega(t)$ is developed when the function $\Omega(t)$ is given by

$$\Omega(t) = At^{\alpha}(1 + A_1 t^n). \tag{1}$$

Here A and A_1 are non-negative constants; $\alpha \geqslant 0$ and n is an integer. $U(x)$ and V (= constant) done the potential flow about the body in the steady state. In many practical problems the main-stream velocity may be given in the above separable form, from theoretical and or experimental considerations.

The method of similarity is used here to solve the unsteady three dimensional boundary layer equations and velocity field together with some boundary layer characteristics are obtained in analytical form. The theoretical basis for this method is a generalization of a result of Slavtchev [1] for unsteady two-dimensional boundary layer flow.

It should be mentioned that this problem has attracted the attention of a number of authors over the years [2], [3], [4], [5].

2. Boundary layer equations

The Cartesian co-ordinate system is employed with the x-axis measured along the body surface perpendicular to the generators, y-axis parallel to the generators and z perpendicular to x and y. Thus all physical quantities that describe the flow are independent of y [6]. This simplifies the equations, and in fact the motion in the xz plane is the same as in two-dimensional flow. With this co-ordinate system the governing equations for the flow are the classical boundary layer equations. These equations, stated in reference [2], in dimensional variables aie

$$
\begin{gathered}
\frac{\partial u}{\partial x}+\frac{\partial w}{\partial y}=0, \\
\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+w\frac{\partial u}{\partial z}=\frac{\partial u_e}{\partial t}+u_e\frac{\partial u_e}{\partial x}+\nu\frac{\partial^2 u}{\partial z^2}, \\
\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+w\frac{\partial v}{\partial z}=\frac{\partial v_e}{\partial t}+\nu\frac{\partial^2 v}{\partial z^2},
\end{gathered}
\tag{2}
$$

where (u, v, w) are the velocity components; $u_e(x, t)$ and $v_e(t)$ are the main-stream velocities; t is te time; ν is the kinematic viscosity.

The initial and boundary conditions are simply

$$
\begin{array}{lll}
t=0: & u=u_e(x, t), \quad v=v_e(t) & \text{everywhere}, \\
t>0: & u=v=w=0 & \text{at } \quad z=0, \\
 & u\to u_e(x, t), \quad v\to v_e(t) & \text{as } \quad z\to\infty.
\end{array}
\tag{3}
$$

To solve the equations (2) it is convenient to write u, v, and w as

$$
u=U\Omega\frac{\partial\psi}{\partial\eta}, \quad w=-2\sqrt{\nu t}\,\Omega\frac{dU}{dx}\psi, \quad v=V\Omega\varphi,
\tag{4}
$$

where $\eta=z/2\sqrt{\nu t}$ is the familiar similarity variable of the viscous boundary layer theory. With the new independent variables given by (4) the governing equations (2) reduce identically to

$$
\begin{gathered}
\frac{1}{4}\frac{\partial^3\psi}{\partial\eta^3}+\frac{1}{2}\eta\frac{\partial^2\psi}{\partial\eta^2}+q(t)\left(1-\frac{\partial\psi}{\partial\eta}\right)-t\frac{\partial^2\psi}{\partial t\,\partial\eta}+ \\
+\Omega t\frac{dU}{dx}\left[1+\psi\frac{\partial^2\psi}{\partial\eta^2}-\left(\frac{\partial\psi}{\partial\eta}\right)^2\right]+\Omega tU\left(\frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial\eta^2}-\frac{\partial\psi}{\partial\eta}\frac{\partial^2\psi}{\partial x\,\partial\eta}\right)=0, \\
\frac{1}{4}\frac{\partial^2\varphi}{\partial\eta^2}+\frac{1}{2}\eta\frac{\partial\varphi}{\partial\eta}+q(t)(1-\varphi)-t\frac{\partial\varphi}{\partial t}+\Omega t\frac{dU}{dx}\varphi\frac{\partial\varphi}{\partial\eta}+ \\
+\Omega tU\left(\frac{\partial\psi}{\partial x}\frac{\partial\varphi}{\partial\eta}-\frac{\partial\psi}{\partial\eta}\frac{\partial\varphi}{\partial x}\right)=0.
\end{gathered}
\tag{5}
$$

The boundary conditions now become

$$
\begin{array}{ll}
\psi=\frac{\partial\psi}{\partial\eta}=\varphi=0 & \text{at } \quad \eta=0, \\
\frac{\partial\psi}{\partial\eta}, \quad \varphi\to 1 & \text{as } \quad \eta\to\infty.
\end{array}
\tag{6}
$$

In (5) we have put

$$q(t)=\frac{\dot{\Omega} t}{\Omega}=\alpha+\frac{nA_1 t^n}{1+A_1 t}, \tag{7}$$

where the dot refers to a derivative with respect to t. We assume $q \neq \alpha$ $(A_1>0)$.

Now we introduce the parameters

$$p_k=U^{k-1}\frac{d^k U}{dx^k}\Omega^k t^k, \quad k=1, 2, 3, \ldots \tag{8}$$

and the following properties are used

$$\begin{gathered}
U\Omega t\frac{\partial p_k}{\partial t}=p_{k+1}+(k-1)p_1 p_k, \quad t\frac{\partial p_k}{\partial t}=(1+q)kp_k, \\
t\frac{\partial}{\partial t}=t\frac{\partial}{\partial t}+\sum_{k=1}^{\infty} t\frac{\partial p_k}{\partial t}\frac{\partial}{\partial p_k}=t\frac{\partial}{\partial t}+(1+q)\sum_{k=1}^{\infty} kp_k\frac{\partial}{\partial p_k}, \\
\Omega t U\frac{\partial}{\partial x}=\sum_{k=1}^{\infty} U\Omega t\frac{\partial p_k}{\partial x}\frac{\partial}{\partial p_k}=\sum_{k=1}^{\infty}[p_{k+1}+(k-1)p_1 p_k]\frac{\partial}{\partial p_k}.
\end{gathered} \tag{9}$$

With the aid of (8) and (9) the equations (5) can be written as

$$\begin{aligned}
&\frac{1}{4}\frac{\partial^3\psi}{\partial\eta^3}+\frac{1}{2}\eta\frac{\partial^2\psi}{\partial\eta^2}+q\left(1-\frac{\partial\psi}{\partial\eta}\right)-t\frac{\partial^2\psi}{\partial t\,\partial\eta}- \\
&\quad -(1+q)\sum_{k=1}^{\infty} kp_k\frac{\partial^2\psi}{\partial\eta\partial p_k}+p_1\left[1+\psi\frac{\partial^2\psi}{\partial\eta^2}-\left(\frac{\partial\psi}{\partial\eta}\right)^2\right]+ \\
&\quad +\sum_{k=1}^{\infty}[p_{k+1}+(k-1)p_1 p_k]\left(\frac{\partial\psi}{\partial p_k}\frac{\partial^2\psi}{\partial\eta^2}-\frac{\partial\psi}{\partial\eta}\frac{\partial^2\psi}{\partial\eta\partial p_k}\right)=0, \\
&\frac{1}{4}\frac{\partial^2\varphi}{\partial\eta^2}+\frac{1}{2}\eta\frac{\partial\varphi}{\partial\eta}+q(1-\varphi)-t\frac{\partial\varphi}{\partial t}-(1+q)\sum_{k=1}^{\infty} kp_k\frac{\partial\varphi}{\partial p_k}+ \\
&\quad +\sum_{k=1}^{\infty}[p_{k+1}+(k-1)p_1 p_k]\left(\frac{\partial\psi}{\partial p_k}\frac{\partial\varphi}{\partial\eta}-\frac{\partial\psi}{\partial\eta}\frac{\partial\varphi}{\partial p_k}\right)=0,
\end{aligned} \tag{10}$$

subjected to the boundary conditions (6).

3. Solution of the problem

The problem of determination of the velocity field will now be solved if equations (10) are solved. For such purposes series solutions to partial differential equations (10) are sought in the following forms

$$\begin{aligned}
\psi=&\psi_0(\eta, t)+\psi_1(\eta, t)p_1+\psi_{11}(\eta, t)p_1^2+\psi_2(\eta, t)p_2+ \\
&+\psi_{111}(\eta, t)p_1^3+\psi_{12}(\eta, t)p_1 p_2+\psi_3(\eta, t)p_3+\cdots, \\
\varphi=&\varphi_0(\eta, t)+\varphi_1(\eta, t)p_1+\varphi_{11}(\eta, t)p_1^2+\varphi_2(\eta, t)p_2+ \\
&+\varphi_{111}(\eta, t)p_1^3+\varphi_{12}(\eta, t)p_1 p_2+\varphi_3(\eta, t)p_3+\cdots.
\end{aligned} \tag{11}$$

Substituting for ψ and φ into (10) and equating the coefficients of like powers and combinations of p_k we obtain the following set of equations

$$
\begin{aligned}
&\frac{1}{4}\frac{\partial^3\psi_0}{\partial\eta^3}+\frac{1}{2}\eta\frac{\partial^2\psi_0}{\partial\eta^2}+q\left(1-\frac{\partial\psi_0}{\partial\eta}\right)-t\frac{\partial^2\psi_0}{\partial t\,\partial\eta}=0,\\
&\frac{1}{4}\frac{\partial^2\varphi_0}{\partial\eta^2}+\frac{1}{2}\eta\frac{\partial\varphi_0}{\partial\eta}+q(1-\varphi_0)-t\frac{\partial\varphi_0}{\partial t}=0,
\end{aligned}
\tag{12}
$$

with the boundary conditions

$$
\begin{aligned}
&\psi_0=\frac{\partial\psi_0}{\partial\eta}=\varphi_0=0 && \text{at}\quad \eta=0,\\
&\frac{\partial\psi_0}{\partial\eta},\quad \varphi_0\to 1 && \text{as}\quad \eta\to\infty;
\end{aligned}
\tag{13}
$$

$$
\begin{aligned}
&\frac{1}{4}\frac{\partial^3\psi_1}{\partial\eta^3}+\frac{1}{2}\eta\frac{\partial^2\psi_1}{\partial\eta^2}-(1+2q)\frac{\partial\psi_1}{\partial\eta}-t\frac{\partial^2\psi_1}{\partial t\,\partial\eta}=\\
&\qquad\qquad=-1+\left(\frac{\partial\psi_0}{\partial\eta}\right)^2-\psi_0\frac{\partial^2\varphi_0}{\partial\eta^2},\\
&\frac{1}{4}\frac{\partial^2\varphi_1}{\partial\eta^2}+\frac{1}{2}\eta\frac{\partial\varphi_1}{\partial\eta}-(1+2q)\varphi_1-t\frac{\partial\varphi_1}{\partial t}=-\psi_0\frac{\partial\varphi_0}{\partial\eta},
\end{aligned}
\tag{14}
$$

with the boundary conditions

$$
\begin{aligned}
&\psi_1=\frac{\partial\psi_1}{\partial\eta}=\varphi_1=0 && \text{at}\quad \eta=0,\\
&\frac{\partial\psi_1}{\partial\eta},\quad \varphi_1\to 0 && \text{as}\quad \eta\to\infty;
\end{aligned}
\tag{15}
$$

and so on.

Now, taking into account that

$$
tq+q(q-\alpha)=(n+\alpha)(q-\alpha),
\tag{16}
$$

functions ψ_k and φ_k can be split up as folows

$$
\begin{aligned}
&\psi_0=F_0(\eta)+(q-\alpha)F_n(\eta),\quad \varphi_0=G_0(\eta)+(q-\alpha)G_n(\eta),\\
&\psi_1=F_0^{(1)}(\eta)+(q-\alpha)F_n^{(1)}(\eta)+(q-\alpha)^2F_n^{(2)}(\eta),\\
&\varphi_1=G_2^{(1)}(\eta)+(q-\alpha)G_n^{(1)}(\eta)+(q-\alpha)^2G_n^{(2)}(\eta).
\end{aligned}
\tag{17}
$$

When these equations, with (16) in mind, are substituted in (12) and (14), and terms in like powers of $(q-\alpha)$ are collected, we obtain the following ordinary equations for $F^{(k)}$ and $G^{(k)}$

$$
\begin{aligned}
&F_0''' + 2\eta F_0'' + 4\alpha(1+F_0') = 0, \quad G_0'' + 2\eta G_0' + 4\alpha(1+G_0) = 0,\\
&F_n''' + 2\eta F_n'' - 4(n+\alpha)F_n' = 4(F_0' - 1), \quad G_n'' + 2\eta G_n' - 4(n+\alpha)G_n = 4(G_0 - 1),\\
&F_0^{(1)'''} + 2\eta F_0^{(1)''} - 4(2\alpha+1)F_0^{(1)'} = -4(1 - F_0'^2 + F_0F_0''),\\
&G_0^{(1)''} + 2\eta G_0^{(1)'} - 4(2\alpha+1)G_0^{(1)} = -4F_0G_0',\\
&F_n^{(1)'''} + 2\eta F_n^{(1)''} - 4(2\alpha+n+1)F_n^{(1)'} = 4(2F_0^{(1)'} - F_0F_n'' - F_0''F_n + 2F_0'F_n'),\\
&G_n^{(1)''} + 2\eta G_n^{(1)'} - 4(2\alpha+n+1)G_n^{(1)} = 4(2G_0^{(1)} - G_0'F_n - F_0G_n'),\\
&F_n^{(2)'''} + 2\eta F_n^{(2)''} - 4(2\alpha+2n+1)F_n^{(2)'} = 4(F_n^{(1)'} - F_nF_n'' + F_n'^2),\\
&G_n^{(2)''} + 2\eta G_n^{(2)'} - 4(2\alpha+2n+1)G_n^{(2)} = 4(G_n^{(1)} - F_nG_n'),
\end{aligned}
\tag{18}
$$

where primes refer to derivates with respect to η. Boundary conditions, according to equations (13), (15) and (17), may now be written as

$$
\begin{aligned}
&F_0(0) = F_0'(0) = G_0(0) = F_n(0) = F_n'(0) = G_n(0) = 0,\\
&F_0'(\infty),\ G_0(\infty) \to 1, \quad F_n'(\infty),\ G_n(\infty) \to 0,\\
&F_0^{(1)}(0) = F_0^{(1)'}(0) = G_0^{(1)}(0) = F_n^{(1)}(0) = F_n^{(1)'}(0) = F_n^{(2)}(0) =\\
&\qquad = F_n^{(2)'}(0) = G_n^{(1)}(0) = G_n^{(2)}(0) = 0,\\
&F_0^{(1)'}(\infty),\ F_n^{(1)'}(\infty),\ F_n^{(2)'}(\infty),\ G_0^{(1)}(\infty),\ G_n^{(1)}(\infty),\ G_n^{(2)}(\infty) \to 0.
\end{aligned}
\tag{19}
$$

Now we consider two particular cases which are most important in view of practical problems, namely

$$
\text{(i)}\ \Omega(t) = A(1 + A_1 t) \quad \text{and} \quad \text{(ii)}\ \Omega(t) = At(1 + A_1 t). \tag{20}
$$

In the case (i) the solutions of (18) subjected to (19) are

$$
F_0' = G_0 = \operatorname{erf}\eta, \quad F_0 = \eta \operatorname{erf}\eta + \frac{1}{\sqrt{\pi}}(e^{-\eta^2} - 1),
$$

$$
F_1' = G_1 = 2\eta^2 \operatorname{erf}\eta + \frac{2}{\sqrt{\pi}}\eta e^{-\eta^2} - 2\eta^2,
$$

$$
F_1 = \frac{2}{3}\eta^3 \operatorname{erf}\eta + \frac{1}{3\sqrt{\pi}}(2\eta^2 - 1)e^{-\eta^2} + \frac{1}{3}\left(\frac{1}{\sqrt{\pi}} - 2\eta^3\right),
$$

$$
\begin{aligned}
F_0^{(1)'} = &-\left(1 + \frac{2}{3\pi}\right)(1 + 2\eta^2) + \left(\frac{1}{2} + \frac{2}{3\pi}\right)\Bigg[(1 + 2\eta^2)\operatorname{erf}\eta +\\
&+ \frac{2}{\sqrt{\pi}}\eta e^{-\eta^2}\Bigg] + \left(\eta^2 - \frac{1}{2}\right)\operatorname{erf}^2\eta + \frac{3}{\sqrt{\pi}}\eta e^{-\eta^2}\operatorname{erf}\eta +\\
&\qquad + \frac{2}{\pi}e^{-2\eta^2} - \frac{4}{3\pi}e^{-\eta^2} + 1,
\end{aligned}
$$

$$G_0^{(1)} = \frac{4}{3\pi}(1+2\eta^2) + \left(\frac{1}{2} - \frac{4}{3\pi}\right)\left[(1+2\eta^2)\,\mathrm{erf}\,\eta + \frac{2}{\sqrt{\pi}}\,\eta\, e^{-\eta^2}\right] -$$

$$-\left(\frac{1}{2}+\eta^2\right)\mathrm{erf}^2\,\eta - \frac{1}{\sqrt{\pi}}\,\eta\, e^{-\eta^2}\,\mathrm{erf}\,\eta - \frac{4}{3\pi}\,e^{-\eta^2},$$

$$F_1^{(1)\prime} = -\left(\frac{1}{3} - \frac{8}{45\pi}\right)(3+12\eta^2+4\eta^4) + \left(\frac{1}{4} - \frac{8}{45\pi}\right)\Big[(3+12\eta^2+$$

$$+4\eta^4)\,\mathrm{erf}\,\eta + \frac{2}{\sqrt{\pi}}(5\eta+2\eta^3)\,e^{-\eta^2}\Big] + \Big(\frac{1}{4} - \eta^2 +$$

$$+\frac{1}{3}\eta^4\Big)\mathrm{erf}^2\,\eta + \Big[\frac{1}{\sqrt{\pi}}\left(\eta^3 - \frac{13}{6}\eta\right)e^{-\eta^2} - 2\left(1+\frac{4}{3\pi}\right)\eta^2 -$$

$$-\frac{8}{3\sqrt{\pi}}\,\eta - \left(1+\frac{4}{3\pi}\right)\Big]\mathrm{erf}\,\eta + \frac{2}{3\pi}(\eta^2-2)\,e^{-2\eta^2} +$$

$$+\frac{1}{3\sqrt{\pi}}\left[\frac{8}{5\sqrt{\pi}} - \left(\frac{21}{2} + \frac{8}{\pi}\right)\eta - \eta^3\right]e^{-\eta^2} + 4\left(1+\frac{2}{3\pi}\right)\eta^2 +$$

$$+\frac{8}{3\sqrt{\pi}}\,\eta + 1 + \frac{4}{3\pi}, \tag{21}$$

$$G_1^{(1)} = \frac{16}{15\pi}(3+12\eta^2+4\eta^4) + \left(\frac{7}{12} - \frac{16}{15\pi}\right)\Big[(3+12\eta^2+$$

$$+4\eta^4)\,\mathrm{erf}\,\eta + \frac{2}{\sqrt{\pi}}(5\eta+2\eta^3)\,e^{-\eta^2}\Big] +$$

$$+\left(\frac{1}{4} - \eta^2 - \frac{7}{3}\eta^4\right)\mathrm{erf}^2\,\eta - \Big[\frac{1}{3\sqrt{\pi}}\left(\frac{5}{2}\eta + 13\eta^3\right)e^{-\eta^2} +$$

$$+2\left(3 - \frac{8}{3\pi}\right)\eta^2 + \frac{8}{3\sqrt{\pi}}\,\eta + 2\left(1 - \frac{4}{3\pi}\right)\Big]\mathrm{erf}\,\eta - \frac{2}{\pi}\,\eta^2 e^{-2\eta^2} +$$

$$+\frac{1}{3\sqrt{\pi}}\left[\frac{8}{5\sqrt{\pi}} - \left(\frac{37}{2} - \frac{16}{\pi}\right)\eta - \eta^3\right]e^{-\eta^2} - \frac{16}{3\pi}\,\eta^2 +$$

$$+\frac{8}{3\sqrt{\pi}}\,\eta - \frac{8}{3\pi}.$$

$$F_1^{(2)\prime} = -\frac{8}{45\pi}\left(1+6\eta^2+4\eta^4+\frac{8}{15}\eta^6\right)-\frac{1}{3}\left(\frac{5}{4}-\frac{8}{15\pi}\right)\left[\left(1+6\eta^2+\right.\right.$$

$$\left.+4\eta^4+\frac{8}{15}\eta^6\right)\operatorname{erf}\eta+\frac{8}{15\sqrt{\pi}}\left.(\eta^5+7\eta^3+\frac{33}{4}\eta\right)e^{-\eta^2}\Big]+$$

$$+\left(\frac{1}{2}\eta^2+\frac{2}{9}\eta^6\right)\operatorname{erf}^2\eta+\left[\frac{1}{\sqrt{\pi}}\left(\frac{1}{2}\eta-\frac{5}{9}\eta^3+\frac{4}{9}\eta^5\right)e^{-\eta^2}+\right.$$

$$+\frac{1}{3}\left(5+\frac{32}{15\pi}\right)\eta^4+2\left(1+\frac{26}{15\pi}\right)\eta^2+\frac{8}{5\sqrt{\pi}}\eta+$$

$$\left.+\frac{5}{12}+\frac{6}{6\pi}\right]\operatorname{erf}\eta+\frac{1}{9\pi}(2-5\eta^2)e^{-2\eta^2}+\frac{1}{\sqrt{\pi}}\left[\frac{7}{3}\left(1+\frac{4}{3\pi}\right)\eta+\right.$$

$$\left.+2\left(1+\frac{16}{45\pi}\right)\eta^3+\frac{4}{5\sqrt{\pi}}\right]e^{-\eta^2}-\frac{32}{45\pi}\eta^4-\frac{52}{15\pi}\eta^2-\frac{8}{5\sqrt{\pi}}\eta-\frac{6}{5\pi},$$

$$G_1^{(2)} = -\frac{2}{3}\left(1-\frac{8}{5\pi}\right)\left(1+6\eta^2+4\eta^4+\frac{8}{15}\eta^6\right)+$$

$$+\frac{1}{3}\left(\frac{23}{4}-\frac{16}{5\pi}\right)\left[\left(1+6\eta^2+4\eta^4+\frac{8}{15}\eta^6\right)\operatorname{erf}\eta+\right.$$

$$\left.+\frac{8}{15\sqrt{\pi}}\left(\eta^5+7\eta^3+\frac{33}{4}\eta\right)e^{-\eta^2}\right]+\left(\frac{1}{2}\eta^2-\frac{2}{3}\eta^6\right)\operatorname{erf}^2\eta+$$

$$+\left[\frac{1}{\sqrt{\pi}}\left(\frac{1}{2}\eta+\frac{1}{3}\eta^3-\frac{4}{3}\eta^5\right)e^{-\eta^2}-\frac{1}{3}\left(23-\frac{64}{5\pi}\right)\eta^4-\right.$$

$$\left.-4\left(3-\frac{38}{15\pi}\right)\eta^2+\frac{8}{5\sqrt{\pi}}\eta-\frac{23}{12}+\frac{28}{15\pi}\right]\operatorname{erf}\eta+$$

$$+\frac{1}{3\pi}(\eta^2-2\eta^4)e^{-2\eta^2}-\frac{1}{\sqrt{\pi}}\left[\frac{2}{3}\left(11-\frac{32}{5\pi}\right)\eta^3+\right.$$

$$\left.+\left(\frac{33}{4}-\frac{184}{15\pi}\right)\eta-\frac{4}{5\sqrt{\pi}}\right]e^{-\eta^2}+\frac{8}{3}\left(1-\frac{8}{5\pi}\right)\eta^4+$$

$$+4\left(1-\frac{38}{15\pi}\right)\eta^2-\frac{8}{5\sqrt{\pi}}\eta+2\left(\frac{1}{3}-\frac{14}{15\pi}\right),$$

where erf η is the standard error function defined by

$$\operatorname{erf}\eta=\frac{2}{\sqrt{\pi}}\int_0^{\eta}e^{-s^2}\,ds. \tag{22}$$

From (21) we have

$$F_0''(0)=G_0'(0)=\frac{2}{\sqrt{\pi}},\quad F_1''(0)=G_1'(0)=\frac{2}{\sqrt{\pi}},$$

$$F_0^{(1)''}(0)=\frac{4}{\sqrt{\pi}}\left(\frac{1}{2}+\frac{2}{3\pi}\right),\quad G_0^{(1)'}(0)=\frac{4}{\sqrt{\pi}}\left(\frac{1}{2}-\frac{4}{3\pi}\right),$$

(23)
$$F_1^{(1)''}(0)=\frac{1}{\sqrt{\pi}}\left(\frac{7}{6}-\frac{368}{45\pi}\right),\quad G_1^{(1)'}(0)=\frac{1}{\sqrt{\pi}}\left(\frac{11}{6}-\frac{32}{5\pi}\right),$$

$$F_1^{(2)''}(0)=\frac{1}{\sqrt{\pi}}\left(-\frac{11}{10}+\frac{1496}{225\pi}\right),\quad G_1^{(2)'}(0)=\frac{2}{\sqrt{\pi}}\left(-\frac{17}{24}+\frac{334}{75\pi}\right).$$

In the case (ii) the solutions of (18) are

$$F_0'=G_0=1-4g_1,\quad F_0=\eta+4g_{3/2}-\frac{1}{2\Gamma},$$

$$F_1'=G_1=4g_1-32g_2,\quad F_1=32g_{5/2}-4g_{3/2}+\frac{1}{10\Gamma},$$

$$F_0^{(1)}=\frac{2}{3}g_0-\frac{4}{5\Gamma}g_{1/2}-32g_1g_2+32g_{3/2}^2-32\left(5-\frac{6}{5\Gamma^2}\right)g_3,$$

$$G_0^{(1)}=\frac{2}{3}g_0-\frac{4}{5\Gamma}g_{1/2}-4g_1-32g_1g_2+32\left(7+\frac{36}{5\Gamma^2}\right)g_3,$$

$$F_1^{(1)'}=-\frac{5}{6}g_0+\frac{8}{7\Gamma}g_{1/2}+\frac{16}{3}g_1-\frac{32}{5\Gamma}g_{3/2}-32g_2+$$
$$+64\left(5-\frac{6}{5\Gamma^2}\right)g_3-512\left(5-\frac{144}{35\Gamma^2}\right)g_4+64g_1g_2-$$
$$-256g_1g_3+512g_{3/2}g_{5/2}-64g_{3/2}^2-256g_2^2,$$

$$G_1^{(1)}=-\frac{5}{6}g_0+\frac{8}{7\Gamma}g_{1/2}+\frac{32}{3}g_1-\frac{32}{5\Gamma}g_{3/2}-64g_2-$$
$$-64\left(7+\frac{36}{5\Gamma^2}\right)g_3+512\left(15+\frac{96}{7\Gamma^2}\right)g_4+64g_1g_2-$$
(24)
$$-256g_1g_3-256g_2^2,$$

$$F_1^{(2)'}=\frac{1}{6}g_0-\frac{12}{35\Gamma}g_{1/2}-\frac{4}{3}g_1+\frac{96}{35\Gamma}g_{3/2}+\frac{32}{3}g_2-$$
$$-32\left(5-\frac{6}{5\Gamma^2}\right)g_3+512\left(5-\frac{144}{35\Gamma^2}\right)g_4+$$
$$+20480\left(-1+\frac{144}{175\Gamma^2}\right)g_5-512g_{3/2}g_{5/2}-$$
$$-32g_1g_2+256g_1g_3-2048g_2g_3+32g_{3/2}^2+2048g_{5/2}^2+256g_2^2,$$

$$G_1^{(2)} = \frac{1}{6} g_0 - \frac{12}{35\Gamma} g_{1/2} - \frac{8}{3} g_1 + \frac{96}{35\Gamma} g_{3/2} + \frac{64}{3} g_2 +$$

$$+ 32\left(7 + \frac{36}{5\Gamma^2}\right) g_3 - 512\left(15 + \frac{96}{7\Gamma^2}\right) g_4 +$$

$$+ 61440\left(1 + \frac{32}{35\Gamma^2}\right) g_5 - 32 g_1 g_2 + 256 g_1 g_3 - 2048 g_2 g_3 + 256 g_2^2,$$

where $\Gamma(5/2)$ is the usual Gamma function and g_γ is the Gauss' error function or order 2γ defined by

$$g_\gamma(\eta) = \frac{2}{\sqrt{\pi}\,\Gamma(2\gamma+1)} \int_\eta^\infty (s-\eta)^{2\gamma} e^{-s^2}\, ds. \tag{25}$$

Functions (24) give

$$\begin{aligned} &F_0''(0) = G_0'(0) = \frac{4}{\sqrt{\pi}}, \quad F_1''(0) = G_1'(0) = \frac{4}{3\sqrt{\pi}}, \\ &F_0^{(1)''}(0) = \frac{2}{\sqrt{\pi}}\left(\frac{31}{30} - \frac{128}{225\pi}\right), \quad G_0^{(1)'}(0) = \frac{2}{\sqrt{\pi}}\left(\frac{7}{6} - \frac{256}{75\pi}\right), \\ &F_1^{(1)''}(0) = \frac{2}{\sqrt{\pi}}\left(\frac{9}{70} - \frac{12032}{11025\pi}\right), \quad G_1^{(1)'}(0) = \frac{2}{\sqrt{\pi}}\left(\frac{1}{2} - \frac{6656}{11025\pi}\right), \\ &F_1^{(2)''}(0) = \frac{2}{\sqrt{\pi}}\left(-\frac{176}{945} + \frac{22144}{33075\pi}\right), \quad G_1^{(2)'}(0) = \frac{2}{\sqrt{\pi}}\left(-\frac{2}{9} + \frac{70912}{99225\pi}\right). \end{aligned} \tag{26}$$

When these universal functions are known, it then becomes a simple matter to determine the boundary layer characteristics. Thus, the skin friction components at the body surface are

$$\begin{aligned} \tau_x = \frac{1}{2}\rho\, u_e \sqrt{\nu/t}\, \Bigg\{ & F_0''(0) + \frac{A_1 t}{1 + A_1 t} F_1''(0) + \\ & + \frac{dU}{dx} \frac{A}{A_1^{\alpha+1}} \Bigg[A_1^{\alpha+1} t^{\alpha+1} (1 + A_1 t) F_0^{(1)''}(0) + \\ & + A_1^{\alpha+2} t^{\alpha+2} F_1^{(1)''}(0) + \frac{A_1^{\alpha+3} t^{\alpha+3}}{1 + A_1 t} F_1^{(2)''}(0) \Bigg] \Bigg\}, \\ \tau_z = \frac{1}{2}\rho\, v_e \sqrt{\nu/t}\, \Bigg\{ & G_0'(0) + \frac{A_1 t}{1 + A_1 t} G_1'(0) + \\ & + \frac{dU}{dx} \frac{A}{A_1^{\alpha+1}} \Bigg[A_1^{\alpha+1} t^{\alpha+1} (1 + A_1 t) G_2^{(1)'}(0) + \\ & + A_1^{\alpha+2} t^{\alpha+2} G_1^{(1)'}(0) + \frac{A_1^{\alpha+3} t^{\alpha+3}}{1 + A_1 t} G_1^{(2)'}(0) \Bigg] \Bigg\}. \end{aligned} \tag{27}$$

4. Applications

We are now in position to apply (27) for any particular case. We select the plate and the cylinder as convenient examples.

Thus, as is know, for a flat plate inclined with angle β to the horizontal the velocities at infinity are

$$U_\infty = W_0 \cos\beta \quad \text{and} \quad V_\infty = W_0 \sin\beta, \tag{28}$$

where W_0 is a constant.

If we introduce the dimensionaless quantities

$$u = U_\infty u^+, \quad v = U_\infty v^+, \quad z = \frac{b}{\sqrt{R}} z^+, \quad t = \frac{b}{U_\infty} t^+, \quad R = \frac{U_\infty b}{\nu}, \tag{29}$$

where b is the length of the plate and R is the Reynolds number, then the coefficients of the skin friction are

$$C_x = \frac{Cz}{\operatorname{tg}\beta} = \frac{4}{\sqrt{\pi R}} \frac{1}{\sqrt{t}\,(1+Bt)} \left(1 + \frac{Bt}{1+Bt}\right), \tag{30}$$

in the case (i) and

$$C_x = \frac{Cz}{\operatorname{tg}\beta} = \frac{4}{\sqrt{\pi R}} \frac{1}{\sqrt{t}\, Bt\,(1+Bt)} \left[1 + \frac{Bt}{3\,(1+Bt)}\right], \tag{31}$$

in the case (ii), respectively. Here B is a constant $(= A_1 b/U_\infty)$ and the cross has been omitted. These coefficients are plotted againts Bt in Fig. 1.

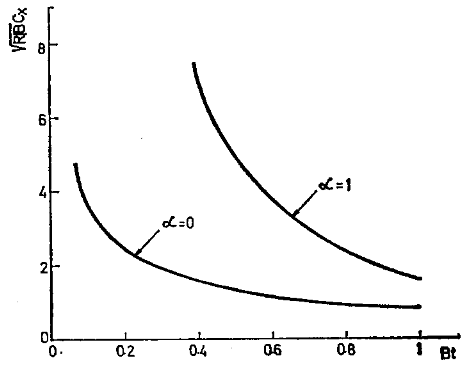

Fig. 1. Variation of C_x with Bt.

For a circular cylinder of radius a we have

$$U(x) = 2U_\infty \sin\frac{x}{a}, \tag{32}$$

and (23) become

$$\tau_x = 2\rho U_\infty \sin\frac{x}{a}\sqrt{\nu/\pi t}\left\{1+\frac{A_1 t}{1+A_1 t}+\frac{A}{aA_1}\cos\frac{x}{a}\left[2\cdot 849 A_1 t(1+A_1 t)-\right.\right.$$

$$\left.\left.-1\cdot 328 A_1^2 t^2 + 1\cdot 003\frac{A_1^3 t^3}{1+A_1 t}\right]\right\}, \tag{33}$$

$$\tau_z = \rho V_\infty \sqrt{\nu/\pi t}\left\{1+\frac{A_1 t}{1+A_1 t}+\frac{A}{aA_1}\cos\frac{x}{a}\left[0\cdot 304 A_1 t(1+A_1 t)-\right.\right.$$

$$\left.\left.-0\cdot 205 A_1^2 t^2 + 1\cdot 922\frac{A_1^3 t^3}{1+A_1 t}\right]\right\},$$

in the case (i) and

$$\tau_x = 4U_\infty \sin\frac{x}{a}\sqrt{\nu/\pi t}\left\{1+0\cdot 333\frac{A_1 t}{1+A_1 t}+\right.$$

$$\left.+\frac{A}{aA_1^2}\cos\frac{x}{a}\left[0\cdot 852 A_1^2 t^2(1+A_1 t)-0\cdot 219 A_1^3 t^3+0\cdot 027\frac{A_1^4 t^4}{1+A_1 t}\right]\right\}, \tag{34}$$

$$\tau_z = 2\rho V_\infty \sqrt{\nu/\pi t}\left\{1+0\cdot 333\frac{A_1 t}{1+A_1 t}+\frac{A}{aA_1^2}\cos\frac{x}{a}\left[0\cdot 079 A_1^2 t^2(1+A_1 t)+\right.\right.$$

$$\left.\left.+0\cdot 308 A_1^3 t^3 + 0\cdot 005\frac{A_1^4 t^4}{1+A_1 t}\right]\right\},$$

in the case (ii), respectively.

In accordance with the usual practice in unsteady boundary layer, we are interested in finding the position of separation of forward flow from the contour and therefore the time t_s at which separation occurs. This position is given by $\tau_x = 0$ and separation corresponds to having $\cos x/a = -1$, i.e. the last stagnation point. According to equations (33) and (34), the time of separation is given by

$$1+\frac{A_1 t_s}{1+A_1 t_s}-\lambda\left[2\cdot 849 A_1^2 t_s^2(1+A_1 t_s)-\right.$$

$$\left.-1\cdot 438 A_1^3 t_s^3 + 1\cdot 003\frac{A_1^3 t_s^3}{1+A_1 t_s}\right]=0, \tag{35}$$

in the case (i) and

$$1+0\cdot 333\frac{A_1 t_s}{1+A_1 t_s}-\lambda\left[0\cdot 852 A_1^2 t_s^2(1+A_1 t_s)-\right.$$

$$\left.-0\cdot 219 A_1^3 t_s^3 + 0\cdot 027\frac{A_1^4 t_s^4}{1+A_1 t_s}\right]=0, \tag{36}$$

in the case (ii), respectively, where

$$\lambda = \frac{A}{aA_1^{\alpha+1}},\quad \alpha = 0,1. \tag{37}$$

Fig. 2 shows the variation of λ with $A_1 t_s$.

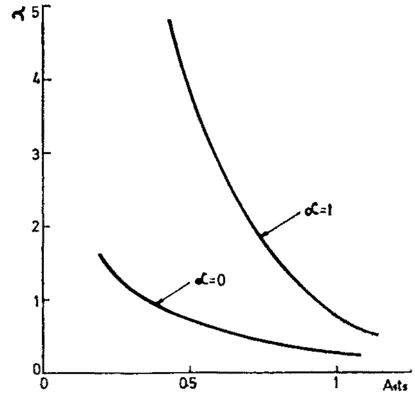

Fig. 2. Dependence of λ by $A_1 t_s$.

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